Fundamental Solutions and Green Functions for Nonhomogeneous Second Order Elliptic Operators with Discontinuous Coefficients

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Abstract

This thesis is devoted to the properties of fundamental solutions, Green functions, and Neumann-Green functions for general non-homogeneous second order elliptic systems with discontinuous coefficients.

We establish existence, uniqueness, and scale-invariant estimates for fundamental solutions of non-homogeneous second order elliptic systems

$$\mathcal{L}\mathbf{u} = -D_\alpha\left(\mathbf{A}^{\alpha\beta}D_\beta\mathbf{u} + \mathbf{b}^\alpha\mathbf{u}\right) + \mathbf{d}^\beta D_\beta\mathbf{u} + \mathbf{V}\mathbf{u},$$

where $\mathbf{A}$ is a matrix of bounded measurable coefficients in $\mathbb{R}^n$ and $\mathbf{b}, \mathbf{d}, \mathbf{V}$ are in suitable integrability classes, as well as for the corresponding Green functions in arbitrary open, connected sets. We impose certain non-homogeneous versions of de Giorgi-Nash-Moser bounds on the weak solutions and investigate in detail the assumptions on the lower order terms sufficient to guarantee such conditions. Our results, in particular, establish the existence and fundamental estimates for the Green functions associated to the Schrödinger $(-\Delta + V)$ and generalized Schrödinger $(-\operatorname{div} A\nabla + V)$ operators with bounded measurable real and complex coefficients on arbitrary domains.

Most of the results above rely on the construction of the averaged fundamental solutions and Green functions with sharp uniform estimates. We also showcase a different approach to Green and Neumann-Green functions via layer potentials which yields, in addition, certain new mapping properties for the Green operators.

A substantial portion of the results of this thesis gave rise to [14], submitted for publication.

Contents

Chapter 1

Introduction

In the present work, we initiate the study of the well-posedness of boundary value problems with boundary data in L^p for non-homogeneous second order uniformly elliptic systems, formally given by

$$\mathcal{L}\mathbf{u} = -D_\alpha \left(\mathbf{A}^{\alpha\beta} D_\beta \mathbf{u} + \mathbf{b}^\alpha \mathbf{u} \right) + \mathbf{d}^\beta D_\beta \mathbf{u} + \mathbf{V}\mathbf{u}. \tag{1.0.1}$$

The principal term, $L := -D_\alpha \mathbf{A}^{\alpha\beta} D_\beta$, satisfies the following ellipticity and boundedness conditions

$$\int A_{ij}^{\alpha\beta}(x) D_\beta \phi^j(x)\, D_\alpha \phi^i(x)\, dx \geq \lambda \sum_{i=1}^{N} \sum_{\alpha=1}^{n} \int \left| D_\alpha \phi^i(x) \right|^2 dx$$

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \left| A_{ij}^{\alpha\beta}(x) \right|^2 \leq \Lambda^2,$$

for some $0 < \lambda, \Lambda < \infty$, where the first inequality holds for all $\phi = \left(\phi^1, \ldots, \phi^N\right)$ belonging to an appropriate Hilbert space, and the second inequality holds for all x in the domain. Note that, in particular, equations with complex bounded measurable coefficients fit into this scheme. The lower order coefficients $\mathbf{b}, \mathbf{d}$, and $\mathbf{V}$ are also assumed to be non-smooth (and not necessarily continuous), with precise conditions to be given in the body of the paper.

The past 30 years have seen an enormous amount of activity and great breakthroughs in harmonic analysis related to the study of boundary value problems for homogeneous equations and systems ($\mathbf{b} = \mathbf{d} = 0$, $\mathbf{V} = 0$). Most notably, the Calderón-Zygmund program has been extended to treat layer potentials associated to fairly general elliptic operators on non-smooth domains; development of the theory of Muckenhoupt weights brought novel tools to handle harmonic measure; the celebrated solution of the Kato problem has opened the door to treat complex coefficient operators; and finally, a recent leap in understanding of connections between absolute continuity of harmonic measure and Carleson measure estimates on solutions has changed the landscape in related geometric measure theory and offered a completely new level of understanding of geometric, analytic, and PDE properties of sets.

Given all these successes the reader would perhaps be surprised to learn that virtually no results of this type exist for non-homogeneous elliptic operators and systems (1.0.1). A closer look reveals a deep reason beyond such an omission. An important underlying thread of most of the aforementioned results is finding new clever ways to use various scale-invariant estimates on solutions for a "bootstrapping" or "extrapolation" of small-constant results to the general case of interest. Scale-invariance, however, is always tricky and often hopeless for a non-homogeneous equation. Let us give a very simple example. The Harnack inequality guarantees that any function u that is harmonic and positive in a ball $B = B_r(x)$ has $\sup_B u \leq C \inf_B u$, with a constant C uniform in x and r. If u, however, satisfies a Schrödinger equation $-\Delta u + Vu = 0$, the constant C, roughly speaking, depends exponentially on Vr^2. Such a dependence undermines most, if not all, of the aforementioned methods, and has to be avoided. One encounters many similar problems down the road, but let us first narrow down the discussion a little.

The first step in the study of boundary value problems, and the main objective of the present thesis, is a comprehensive treatise of the fundamental solutions and Green functions associated to non-homogeneous elliptic systems on arbitrary domains. We establish existence, uniqueness, and global scale-invariant estimates for the fundamental solution in $\mathbb{R}^n$ and for the Dirichlet Green function in any connected, open set $\Omega \subset \mathbb{R}^n$, where $n \geq 3$.

The fundamental solutions and Green functions for *homogeneous* second order elliptic systems are fairly well-understood by now. We do not aim to review the vast literature addressing various situations with additional smoothness assumptions on the coefficients of the operator and/or the domain and will rather comment on those works that are most closely related to ours. The analysis of Green functions for operators with bounded measurable coefficients goes back to the early 80's, in which [23] (see also [35] for symmetric operators) studied the case of homogeneous equations ($N = 1$) with real coefficients. The case of homogeneous *systems*, and, respectively, equations with complex coefficients, has been treated much more recently in [27] and [30] under the assumptions of local boundedness and Hölder continuity of solutions, the so-called de Giorgi-Nash-Moser estimates. Later on, in [41], the fundamental solution in $\mathbb{R}^n$ was constructed using only the assumption of local boundedness, that is, without the requirement of Hölder continuity. In [8], Barton constructed fundamental solutions, also in $\mathbb{R}^n$ only, in the full generality of homogeneous elliptic systems without assuming any de Giorgi-Nash-Moser estimates. The techniques in [8] are based on descent from the higher order case.

What is not encountered in any of the aforementioned works, and the key difficulty of our study, as mentioned above, is the lack of homogeneity of the system, since this typically results in a lack of scale-invariant bounds. Here, the existence of solutions relies on a coercivity assumption, which controls the lower-order terms, and the validity of the Caccioppoli inequality. Furthermore, following many predecessors (see, e.g., [27], [30]), we require certain quantitative versions of the local boundedness of solutions. This turns out to be a delicate game, however, to impose local conditions which are sufficient for the construction of fundamental solutions and necessary for most prominent non-homogeneous examples. Indeed, they have not been completely well-understood even in the case of real *equations*, due to the same type of difficulties: Solutions to non-homogeneous equations can grow exponentially with the growth of the domain in the absence of a suitable control on the potential $\mathbf{V}$, even if $\mathbf{b} = \mathbf{d} = \mathbf{0}$.

The present paper can be split into three big portions. In the first part, we prove that one can define the fundamental solution and the Green function, and establish global estimates on par with the aforementioned results for homogeneous equations, roughly speaking, if:

1. The bilinear form associated to $\mathcal{L}$ is coercive and bounded in a suitable Hilbert space.

2. The Caccioppoli inequality holds:

   If $\mathbf{u}$ is a weak solution to $\mathcal{L}\mathbf{u} = \mathbf{0}$ in $U \subset \Omega$ and ζ is a smooth cutoff function, then

$$\int |D\mathbf{u}|^2 \zeta^2 \leq C \int |\mathbf{u}|^2 |D\zeta|^2 ,$$

where C is independent of the subdomain U.

3. The interior scale-invariant Moser bounds hold:

   If $\mathbf{u}$ is a weak solution to $\mathcal{L}\mathbf{u} = \mathbf{f}$ in $B_R \subset \Omega$, for some $R > 0$, where $\mathbf{f} \in L^\ell (B_R)^N$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$, then for any $q > 0$,

$$\sup_{B_{R/2}} |\mathbf{u}| \leq C \left[\left(\fint_{B_R} |\mathbf{u}|^q \right)^{1/q} + R^{2-\frac{n}{\ell}} \, ||\mathbf{f}||_{L^\ell(B_R)} \right],$$

   where C is independent of R.

4. The solutions are Hölder continuous:

   If $\mathbf{u}$ is a weak solution to $\mathcal{L}\mathbf{u} = \mathbf{0}$ in $B_{R_0} \subset \Omega$, for some $R_0 > 0$, then there exists $\eta \in (0,1)$, depending on R_0, and $C_{R_0} > 0$ so that whenever $0 < R \leq R_0$,

$$\sup_{x,y \in B_{R/2}, x \neq y} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|}{|x-y|^\eta} \leq C_{R_0} R^{-\eta} \left(\fint_{B_R} |\mathbf{u}|^{2^*} \right)^{1/2^*}.$$

If, in addition, the boundary scale-invariant Moser bounds hold (that is, the Moser estimate holds for solutions with trace zero on balls possibly intersecting the boundary), then the Green functions exhibit respectively stronger boundary estimates. This part of the paper is modeled upon the work in [27] and [30]. However, the scaling issues and identifying the exact form of necessary conditions that are compatible with the principal non-homogeneous examples make our arguments considerably more delicate. Note, in particular, the local nature of Hölder estimates versus the global nature of Moser-type bounds. The Moser-type bounds are independent of the domain, whereas the Hölder estimates may depend on the size of the ball. The verification of local bounds and Hölder continuity in our arguments follows a traditional route (see [22], [24], [47]). However, we have to carefully adjust the arguments so that the dependence on constants coincides with our constructions of fundamental solutions.

In the second portion of this work, we motivate the assumptions above by showing that conditions (1)–(4) above are valid in the following three situations:

Case 1. *Homogeneous operators*: $\mathbf{b}, \mathbf{d}, \mathbf{V} \equiv \mathbf{0}$ and the function space for solutions is $\mathbf{F}(\Omega) = Y^{1,2}(\Omega)^N$. Here, $Y^{1,2}(\Omega)$ is the family of all weakly differentiable functions $u \in L^{2^*}(\Omega)$, with $2^* = \frac{2n}{n-2}$, whose weak derivatives are functions in $L^2(\Omega)$.

Case 2. *Lower order coefficients in L^p*: There exist $p \in \left(\frac{n}{2}, \infty\right]$, $s, t \in (n, \infty]$ so that $\mathbf{V} \in L^p(\Omega)^{N \times N}$, $\mathbf{b} \in L^s(\Omega)^{n \times N \times N}$, $\mathbf{d} \in L^t(\Omega)^{n \times N \times N}$ and we take the function space for solutions to be $\mathbf{F}(\Omega) = W^{1,2}(\Omega)^N$. As usual, $W^{1,2}(\Omega)$ is the family of all weakly differentiable functions $u \in L^2(\Omega)$ whose weak derivatives are functions in $L^2(\Omega)$. The lower-order terms are chosen so that the bilinear form associated to $\mathcal{L}$ is coercive. For conditions (3)-(4), we assume further that $\mathbf{V} - \operatorname{div} \mathbf{b} \geq 0$ and $\mathbf{V} - \operatorname{div} \mathbf{d} \geq 0$ in the sense of distributions.

Case 3: *Reverse Hölder potentials*: $\mathbf{V} \in B_p$, the reverse Hölder class, for some $p \in \left[\frac{n}{2}, \infty\right)$, $\mathbf{b}, \mathbf{d} \equiv \mathbf{0}$, and $\mathbf{F}(\Omega) = W_V^{1,2}(\Omega)^N$, a weighted Sobolev space (with the weight given by a certain maximal function associated to $\mathbf{V}$ – see Section 2.3.3 for the definitions).

To be precise, we show that in each case listed above, (1)–(2) hold for the general systems, while (3)–(4) holds for *equations* and, hence, the resulting estimates on fundamental solutions and Green functions are valid for the *equations* with real coefficients in each of the three cases.

Most of the results above rely on the construction of the averaged fundamental solutions and Green functions via the Lax-Milgram lemma with sharp uniform estimates. In the last portion of this work we showcase a different approach to Green and Neumann-Green functions, via layer potentials, which yields, in addition, certain new mapping properties for the Green operators. To do so, we restrict the discussion to homogeneous complex coefficient operators with t-independent coefficients in $\mathbb{R}^n_+$ and prove, for instance, endpoint weak-L^p result for the derivatives of the Green function and the Neumann-Green function (the Green function analogue for the Neumann problem) under the assumption that the respective boundary layer potentials are invertible. In the context of homogeneous operators these results have been simultaneously and independently obtained in [30] for the Green function and in [13] for the Neumann-Green function under a certain local boundedness property for solutions. Both of those papers relied on the methods of [27] rather than layer potentials, and our methods act as a guide for the use of layer potentials in the present context and, moreover, give mapping properties of the layer potential operators that, to our knowledge, are new. We expect that some version of this approach is still available for non-homogeneous operators and would yield new results, but for the moment we have not fully explored such a generalization, primarily because even basic L^2 estimates on non-homogeneous layer potentials are much more challenging than their homogeneous counterparts. This is certainly an important subject for future investigation, and for now the last chapter should be taken as an example of a different, potentially interesting, approach.

The outline of the thesis is as follows:

- Chapter 2 expands a bit the discussion of the history briefly outlined above and otherwise mainly concentrates on a range of basic results for involved function spaces that will be used throughout the thesis.

  An experienced reader can probably skip right to Chapter 3, returning to the results of Chapter 2 as necessary.

- Chapter 3 is the core of the thesis which collects main results: Theorems 3.2.6 and 3.2.10 address the fundamental solution and the Green function, respectively, under certain necessary conditions; Lemmas in Sections 3.3, 3.4, and 3.5 verify the aforementioned necessary conditions in a variety of common situations; Section 3.6 ties this all together and highlights major examples.

- Chapter 4 is devoted to the layer potential approach and concentrates on homogeneous equations in the half-space.

Chapter 2

History and preliminaries

2.1 A detailed historical account

In the present section, we provide a somewhat more detailed historical account of the relevant work on fundamental solutions and Green functions for the homogeneous elliptic operators briefly mentioned in the introduction. Once again, we concentrate on the results for operators with bounded measurable coefficients on general domains and do not mention many important developments under additional smoothness assumptions on the domain and/or the coefficients.

We take $\mathbb{R}^n$, $n \geq 3$, as our ambient space and $\Omega \subset \mathbb{R}^n$ an open, connected (possibly unbounded) set. Recall that, roughly speaking, the fundamental solution, $\Gamma(x, y)$, assocated to an elliptic operator $\mathcal{L}$ is the solution to $\mathcal{L}_x \Gamma(x, y) = \delta_y(x)$ in $\mathbb{R}^n$ and the Green function associated to $\mathcal{L}$ on a domain Ω (again, roughly speaking) is the solution to

$$\begin{cases} \mathcal{L}_x G_D(x, y) = \delta_y(x) & \text{in } \Omega, \\ G_D(x, y) = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $\delta_y(x)$ is the Dirac delta distribution concentrated at $y \in \Omega$ in the variable x. The subscript in $\mathcal{L}_x$ is used to indicate that the operator acts on the x-variable. These, along with the Neumann-Green function, are our main objects of study, and their precise definitions will be given in later chapters.

We begin with the classical result of W. Littman, G. Stampacchia, and H. F. Weinberger ([35]). In that paper, the authors establish that a Green function exists for the homogeneous operator ($L = -\operatorname{div} A\nabla$) with real, elliptic, symmetric, and bounded coefficients, and that the following pointwise bound holds:

$$G(x,y) \leq C|x-y|^{2-n}, \quad \forall x,y \in \Omega, \quad x \neq y.$$

The Neumann-Green function for the symmetric operator on a ball (which in this context is as general as a bounded starlike Lipschitz domain) was studied in [32], and additional estimates were established. These results are summarized in [31, Theorems 1.2.8 and 1.6.3].

The classical result for non-symmetric homogeneous operators (with real, elliptic, and bounded coefficients) comes from M. Grüter and K.-O. Widman ([23]). In that paper, the authors show the existence of a Green function and that the expected pointwise, L^p, and weak-type estimates hold. Specifically, they proved that for any connected, open set $\Omega \subset \mathbb{R}^n$, there exists a locally integrable function $G : \Omega \times \Omega \to \mathbb{R}$ such that

$$\int_\Omega A(x)\nabla_x G(x,y) \cdot \nabla\phi(x)dx = \phi(y) \tag{2.1.1}$$

for all $\phi \in C_c^\infty(\Omega)$, the pointwise estimate

$$G(x,y) \leq C|x-y|^{2-n}, \quad \forall x,y \in \Omega \tag{2.1.2}$$

holds whenever $x \neq y$, and the following estimates hold uniformly in $y \in \Omega$:

$$G(\,\cdot\,,y) \in L^{\frac{n}{n-2},\infty}(\Omega), \tag{2.1.3}$$

$$\nabla G(\,\cdot\,,y) \in L^{\frac{n}{n-1},\infty}(\Omega), \tag{2.1.4}$$

$$G(\,\cdot\,,y) \in W^{1,2}(\Omega \setminus B_r(y)) \cap W_0^{1,1}(\Omega), \quad \forall r > 0 \tag{2.1.5}$$

$$G(\,\cdot\,,y) \in W_0^{1,p}(\Omega), \quad \text{for each } p \in [1, \tfrac{n}{n-1}). \tag{2.1.6}$$

Here, $W^{k,p}$ and $W_0^{k,p}$ indicate the standard Sobolev spaces and $L^{p,\infty}$ is the weak-L^p space given as the set of functions satisfying the estimate

$$|\{x : |f(x)| > s\}| \leq Cs^{-p}, \quad \forall s > 0.$$

The statements (2.1.2)–(2.1.6) are exactly the type of the estimates that we seek here. The methods of [23] relied heavily on Harnack-type inequalities and the maximum principle and were thus limited to real operators and to a single equation.

More recently, S. Hofmann and S. Kim ([27]) established the existence of the Green function (or "Green matrix") of second order elliptic systems with real, elliptic, and bounded coefficients. The key observation of that paper is that what had previously been done by use of Harnack-type inequalities and the maximum principle (e.g., in [23]) can be done in greater generality under the assumption that the operator admits the de Giorgi-Nash oscillation estimate. The authors were thus able to construct the Green function and prove local versions of the standard pointwise, L^p, and weak-type estimates.

They did not, however, derive global estimates as in (2.1.2)-(2.1.6). That was accomplished by K. Kang and S. Kim in [30] (for the Green function) and by J. Choi and S. Kim in [13] (for the Neumann-Green function) under an additional local boundedness assumption on solutions. In fact, the cited authors showed that the global pointwise estimate (2.1.2) for the Green or Neumann-Green function is equivalent to this extra local boundedness condition, provided solutions already enjoy the de Giorgi-Nash estimate of [27]. However, the local boundedness condition does not hold in general. In addition, [13] establishes (without the extra local boundedness assumption) existence and local estimates of the Neumann-Green function which are analogous to the results for the Green function in [27].

Later on, A. Rosen in [41] constructs the fundamental solution using only the assumption of local boundedness and for $n \geq 2$. Finally, A. Barton ([8]) establishes gradient estimates on the fundamental solution in L^2 without any Hölder requirement of solutions by descent from the higher order case.

All this concerns solely the operators without lower order terms. Fundamental solution and Green function estimates for the non-homogeneous operator $\mathcal{L} = -\mathrm{div}\,(A\nabla + b) + d \cdot \nabla + V$ have been significantly less studied. In [47], G. Stampacchia proved existence and pointwise estimates of the Green function under the assumptions that A is symmetric, b, $d \in L^n \cap L^r$, $V \in L^{r/2}$ for some $r > n$, and the following non-degeneracy condition is satisfied:

$$V - \max\{\mathrm{div}\, b, \mathrm{div}\, d\} \geq c > 0. \tag{2.1.7}$$

The author employed the techniques of [35] which, as previously stated, do not extend to non-symmetric operators or systems. Beyond this result of Stampacchia, very few published results exist on the operator $\mathcal{L}$, even in the case $b = d = 0$, and, in particular, we did not find any literature treating non-homogeneous complex coefficients or systems in this context.

One can sometimes establish bounds on the fundamental solutions and Green functions for elliptic boundary problems by an integration of the estimates of the corresponding heat kernels. To do so requires global in time estimates and a suitable form of uniform exponential decay of the heat kernel. However, non-homogeneous equations typically give rise to bounds for a finite time ($0 < t < T$ with constant depending on T), thus creating an obstacle for this approach (cf., e.g., [15], [3], [6], [40]).

There is one notable exception. In [7], one can find global estimates in the case when the coefficients A, b, d, and V are complex, bounded, the leading term ($-\mathrm{div} A\nabla$) satisfies de Giorgi-Nash-Moser estimates, and the underlying operator is H^1-coercive in the sense that the associated bilinear form

$$\mathcal{B}[u,v] := \int_\Omega A\nabla u \cdot \overline{\nabla v} + u\left(b \cdot \overline{\nabla v}\right) + (d \cdot \nabla u)\,\overline{v} + V u \overline{v}, \quad u, v \in H^1(\mathbb{R}^n)$$

satisfies

$$||u||^2_{H^1(\mathbb{R}^n)} \leq \mathcal{B}[u,u], \quad \forall u \in H^1(\mathbb{R}^n).$$

It is proved that the heat kernel satisfies

$$|K(x,t;y,s)| \leq C(t-s)^{-n/2} e^{\frac{-|x-y|^2}{C'(t-s)}}, \quad \forall x, y \in \mathbb{R}^n, \quad \forall t > s. \tag{2.1.8}$$

At a formal level, this implies a pointwise bound for the fundamental solution associated to $\mathcal{L}$ given by

$$|\Gamma(x,y)| = \left| \int_0^\infty K(x,t;y,0)\,dt \right| \leq \int_0^\infty C t^{-n/2} e^{\frac{-|x-y|^2}{C't}}\,dt \leq C|x-y|^{2-n}, \quad \forall x, y \in \mathbb{R}^n. \tag{2.1.9}$$

However, it is not clear that this approach could provide a basis for a unified theory including all of our targeted estimates (e.g., Hölder continuity of the fundamental solution), and it does not shed light on the situation for Green function on domains ([7] treats the heat kernel on $\mathbb{R}^n$ only). Therefore, in this work we have chosen not to pursue this route.

It is worth mentioning that [4] also derives estimates on the heat kernel of the non-homogeneous operator with complex, bounded coefficients, but in the higher-dimensional case they assume that the coefficients of A are Hölder continuous.

Let us say a few words about Case 3, $-\operatorname{div} A\nabla + V$, $V \in B_p$, $p > n/2$. This is the version of the Schrödinger equation that initially interested us the most. With pointwise bounds on the fundamental solution and the Green function (Theorems 3.2.6 and 3.2.10, respectively), as well as basic Moser, Hölder, and Harnack estimates established in our present work, one can now move on to derive the sharp exponential decay of the fundamental solutions in terms of the Agmon distance associated to the maximal function (2.1.11). Indeed, in the case of the Schrödinger operator $-\Delta + V$ or, more generally, $-\mathrm{div} A\nabla + V$, $V \geq 0$, one expects enhanced decay of solutions, in particular of the fundamental solution and the Green and Neumann-Green functions, due to the influence of the positive potential, V. Philosophically, this goes back at least to S. Agmon [1]. However, not many precise results are available, and virtually nothing is known when it comes to domains and/or $A \neq I$.

In [43], Z. Shen considered $-\Delta + V$ for $V > 0$, $V \in L^\infty_{\mathrm{loc}}$, where there exists $C > 0$ such that, for any ball B in $\mathbb{R}^n$,

$$||V||_{L^\infty(B)} \leq C \fint_B V.$$

He proved that

$$|\Gamma(x,y)| \leq \frac{C_k}{(1 + |x-y|\, m(x,V))^k} \cdot \frac{1}{|x-y|^{n-2}}, \tag{2.1.10}$$

for any $k \in \mathbb{N}$, along with related estimates for the Green and Neumann-Green functions on Lipschitz domains, where

$$m(x,V) := \inf_{r>0} \left\{ \frac{1}{r} : \frac{1}{r^{n-2}} \int_{B_r(x)} V(y)\, dy \leq 1 \right\} \tag{2.1.11}$$

is a weight function originating in the work of Z. Shen, which in turn relies on earlier developments by C. Fefferman and D. Phong ([18]). It is clear, already from Agmon's considerations, that the optimal decay should be exponential, and indeed in [46] Shen proved that for the operator $-\Delta + V$, $V \geq 0$, $V \in (RH)_{n/2}$,

$$C_1 e^{-C_2 d(x,y)} |x-y|^{2-n} \leq \Gamma(x,y) \leq C_3 e^{-C_4 d(x,y)} |x-y|^{2-n}, \tag{2.1.12}$$

where $C_i > 0$, $i = 1,2,3,4$ are some positive constants and d is the Agmon distance associated to $m(x,V)$, i.e.,

$$d(x,y) = \inf_\rho \int_0^1 m(\rho(t), V) \left| \left(\frac{d\rho}{dt}\right)(t) \right| dt,$$

with the infimum taken over all paths, ρ, such that $\rho(0) = x$ and $\rho(1) = y$. Shen also proved that

$$(1 + m(x,V)|x-y|)^{\frac{1}{k_0+1}} \lesssim d(x,y) \lesssim (1 + m(x,V)|x-y|)^{k_0+1},$$

for some $k_0 > 0$. This implies that, in particular,

$$\Gamma(x,y) \lesssim e^{-C_5(1+m(x,V)|x-y|)^{\frac{1}{k_0+1}}} |x-y|^{2-n},$$

for some $C_5 > 0$.

Next, in [34], Kurata established the heat kernel bounds for the operator $-\operatorname{div} A\nabla + V$, $V \geq 0$, $V \in (RH)_{n/2}$, which after integration yield the following upper bound on the fundamental solution of L:

$$\Gamma(x,y) \lesssim e^{-C_6(1+m(x,V)|x-y|)^{\frac{2}{2k_0+3}}} |x-y|^{2-n},$$

for some $C_6 > 0$. This is not sharp, as can be seen from Shen's results above, but it covers all operators $-\operatorname{div} A\nabla + V$ with V as above and A with real, symmetric, bounded coefficients. Hölder-type estimates in this context were obtained in [16] and Sobolev-type estimates were proved in [51]. Both Shen's and Kurata's work rely heavily on the assumption of real coefficients, using Harnack's inequality and the maximum principle, and none of the aforementioned exponential estimates was proved for the Green function or Neumann-Green function on domains.

This question of exponential decay in our more generalized context will be addressed in the upcoming work of B. Poggi and S. Mayboroda, [36], along with the corresponding estimates from below, and sharp estimates akin to (2.1.12) will be obtained for $-\operatorname{div} A\nabla + V$, $V \geq 0$, $V \in (RH)_{n/2}$, in particular, improving Kurata's work. Analogous results will be proved for the magnetic Schrodinger operator, $-(\nabla - iB)^2 + V$. We underline, however, that even to define the fundamental solutions in question and speak about a possibility of exponential decay one needed the results of the present manuscript.

2.2 Basics and notation

We take $n \geq 3$ to be the dimension of the ambient space. Ω is taken to be an open (possibly unbounded) subset of $\mathbb{R}^n$. It will be convenient, primarily in Chapter 4, to distinguish the n^{th} coordinate, so we set

$$\mathbb{R}^n := \{(x', t) : x' \in \mathbb{R}^{n-1},\, t \in \mathbb{R}\},$$
$$\mathbb{R}^n_+ := \{(x', t) : x' \in \mathbb{R}^{n-1},\, t \in (0, \infty)\},$$

and $\mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$. We often drop the subscript from an integral when it is clear from the context. We also often denote the ball $B_r(x)$ by B_r when the center, x, is understood from context. We use the following notation for a ball intersected with the domain:

$$\Omega_r(x) := B_r(x) \cap \Omega.$$

We simply write Ω_r when the center of the ball is clear.

We take $N \geq 1$ to be the dimension of the system of differential equations ($N = 1$ corresponding to the case of a single equation). We often drop the dimensional exponent N from our function spaces when it is clear from the context. Unless otherwise specified, bolded functions ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{f}$, etc.) are N-dimensional vector-valued and unbolded functions are scalar-valued. In particular, we will regularly write the components of a vector-valued function as $\mathbf{u} = (u^1, \ldots, u^N)^T$. All functions are assumed to be measurable.

We typically take indices $i, j \in \{1, \ldots, N\}$ to refer to vector components and $\alpha, \beta \in \{1, \ldots, n\}$ to refer to the dimension in the ambient space. The summation convention is used throughout.

The notation D is used for the derivative matrix of a function, with D_α denoting a single derivative. We use $D_\alpha\mathbf{u}$ to denote the vector $\left(D_\alpha u^1, \ldots, D_\alpha u^N\right)^T$. Sometimes ∇ is used in place of D for scalar-valued functions. ∇u or $\nabla_x u$ indicates the full gradient $(D_1 u, \ldots, D_n u)^T$, whereas $\nabla_{||} u$ or $\nabla_{x'} u$ indicates the gradient in the first $n-1$ variables $(D_1 u, \ldots, D_{n-1} u)^T$. Derivatives are meant in the distributional sense unless otherwise stated.

2.3 Function spaces

2.3.1 Function spaces $Y^{1,2}$ and $W^{1,2}$

Let us recall the definitions. Define the space $Y^{1,2}(\Omega)$ as the family of all weakly differentiable functions $u \in L^{2^*}(\Omega)$, with $2^* = \frac{2n}{n-2}$, whose weak derivatives are functions in $L^2(\Omega)$, endowed with the norm

$$||u||_{Y^{1,2}(\Omega)} = ||u||_{L^{2^*}(\Omega)} + ||Du||_{L^2(\Omega)}.$$

Define $Y_0^{1,2}(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ in the $Y^{1,2}(\Omega)$-norm. By the Sobolev inequality,

$$||u||_{L^{2^*}(\Omega)} \leq C\,||Du||_{L^2(\Omega)} \quad \text{for all } u \in Y_0^{1,2}(\Omega). \tag{2.3.1}$$

The bilinear form

$$\langle u, v\rangle_{Y_0^{1,2}(\Omega)} := \int_\Omega D_\alpha u D_\alpha v \tag{2.3.2}$$

defines an inner product on $Y_0^{1,2}(\Omega)$, where the summation is taken over $\alpha = 1, \ldots, n$.

The vector-valued space $Y_0^{1,2}(\Omega)^N$ for $N > 1$ is a Hilbert space with inner product

$$\langle \mathbf{u}, \mathbf{v}\rangle_{Y_0^{1,2}(\Omega)^N} := \int_\Omega D_\alpha u^i D_\alpha v^i, \tag{2.3.3}$$

where the summation is taken over $\alpha = 1, \ldots, n$ and $i = 1, \ldots, N$. The space has corresponding norm

$$||\mathbf{u}||_{Y_0^{1,2}(\Omega)^N} = \langle \mathbf{u}, \mathbf{u}\rangle_{Y_0^{1,2}(\Omega)^N} = ||D\mathbf{u}||_{L^2(\Omega)^N}.$$

Define $W^{1,2}(\Omega)$ to be the usual Sobolev space, i.e., the space of all weakly differentiable functions $u \in L^2(\Omega)$, whose weak derivatives are functions in $L^2(\Omega)$, endowed with the norm

$$||u||_{W^{1,2}(\Omega)} = ||u||_{L^2(\Omega)} + ||Du||_{L^2(\Omega)}.$$

Let $W_0^{1,2}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in the $W^{1,2}$-norm.

This section will explore various connections between W- and Y-spaces. We remark that for any open, connected set, Ω, in $\mathbb{R}^n$, by completeness of $W^{1,2}(\Omega)$ and $Y^{1,2}(\Omega)$,

$$W_0^{1,2}(\Omega) \hookrightarrow W^{1,2}(\Omega) \text{ and } Y_0^{1,2}(\Omega) \hookrightarrow Y^{1,2}(\Omega). \tag{2.3.4}$$

Lemma 2.3.1. *For any open set $\Omega \subset \mathbb{R}^n$*

$$W_0^{1,2}(\Omega) \hookrightarrow Y_0^{1,2}(\Omega).$$

Proof. Let $u \in W_0^{1,2}(\Omega)$. Then there exists $u_i \in C_c^\infty(\Omega)$ such that

$$\lim_{i\to\infty} ||u_i - u||_{W^{1,2}(\Omega)} = 0.$$

By the Sobolev inequality applied to $u_i - u_k$ we have

$$||u_i - u_k||_{L^{2^*}(\Omega)} \leq c_n \, ||Du_i - Du_k||_{L^2(\Omega)} \leq c_n \, ||u_i - u_k||_{W^{1,2}(\Omega)},$$

and therefore, $\{u_i\}_{i=1}^\infty$ is Cauchy in $Y^{1,2}(\Omega)$. Hence, there is a limit in $Y_0^{1,2}(\Omega)$ and since this limit is, in particular, in $L^{2^*}(\Omega)$, it must coincide with u a.e. □

Before stating the next result, we recall a standard smoothing procedure.

Definition 2.3.2. *For any $U \subset \mathbb{R}^n$ open, and any $\varepsilon > 0$, define $U_\varepsilon = \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}$.*

Definition 2.3.3. *Define the function $\phi \in C_c^\infty(\mathbb{R}^n)$ by*

$$\phi(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where the constant $C > 0$ is chosen so that $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$. We refer to ϕ as the standard mollifier.

For every $\varepsilon > 0$, set

$$\phi_\varepsilon(x) = \frac{C}{\varepsilon^n} \phi\left(\frac{x}{\varepsilon}\right).$$

We remark that for every $\varepsilon > 0$, $\phi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp} \phi_\varepsilon \subset B_\varepsilon(0)$ and $\int_{\mathbb{R}^n} \phi_\varepsilon(x)\, dx = 1$.

Definition 2.3.4. *For any function f that is locally integrable in U, we may define*

$$f^\varepsilon := \phi_\varepsilon * f \quad \text{in } U_\varepsilon.$$

That is, for every $x \in U_\varepsilon$,

$$f^\varepsilon(x) = \int_{B_\varepsilon(0)} \phi_\varepsilon(y) f(x-y)\, dy = \int_U \phi_\varepsilon(x-y) f(y)\, dy.$$

The proofs of the first four statements below may be found in the appendix of [17], and the last one is a part of the proof of Theorem 1 in [17], § 5.3.1.

Lemma 2.3.5 (Properties of mollifiers)**.** *Let U be an arbitrary open set in $\mathbb{R}^n$ and let $f \in L^1_{loc}(U)$. Then*

1. $f^\varepsilon \in C^\infty(U_\varepsilon)$.

2. $f^\varepsilon \to f$ *a.e. as* $\varepsilon \to 0$.

3. *If* $f \in C(U)$, *then* $f^\varepsilon \to f$ *uniformly on compact subsets of* U.

4. *If* $1 \le q < \infty$ *and* $f \in L^q_{\text{loc}}(U)$, *then* $f^\varepsilon \to f$ *in* $L^q_{\text{loc}}(U)$.

5. *If, in addition,* f *is weakly differentiable on* U *and* $Df \in L^1_{loc}(U)$, *then*

$$Df^\varepsilon = \phi_\varepsilon * Df \ \text{ in } U_\varepsilon.$$

Lemma 2.3.6. *If* $\Omega = \mathbb{R}^n$, *then we have the following relations:*

$$W^{1,2}_0(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n) \hookrightarrow Y^{1,2}_0(\mathbb{R}^n) = Y^{1,2}(\mathbb{R}^n). \tag{2.3.5}$$

where the inclusion is strict.

Proof. To show that $W^{1,2}_0(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$, we take any $u \in W^{1,2}(\mathbb{R}^n)$, multiply it by a smooth cut-off function ζ_R, for $R > 0$, that is supported in B_{2R} and equal to 1 on B_R, and convolve the product with a standard mollifier ϕ_ε, $\varepsilon > 0$. One can show that $u_{R,\varepsilon} := \phi_\varepsilon * (u\zeta_R) \in C^\infty_c(\mathbb{R}^n)$ converges to u in $L^2(\mathbb{R}^n)$, and that the derivatives converge to Du in $L^2(\mathbb{R}^n)$, as $\varepsilon \to 0$, $R \to \infty$. Indeed, one can see directly from the properties of the Lebesgue integral that u_R belongs to $W^{1,2}(\mathbb{R}^n)$ and converges to u in the $W^{1,2}(\mathbb{R}^n)$-norm since $u \in W^{1,2}(\mathbb{R}^n)$. Now, since each u_R is compactly supported, the fact that $u_{R,\varepsilon}$ converges to u_R as $\varepsilon \to 0$ in L^2 is due to (4) in Lemma 2.3.5. The fact that each $Du_{R,\varepsilon}$ exists and converges to Du_R in $L^2(\mathbb{R}^n)$ follows from a combination of (5) and (4) in Lemma 2.3.5. The same argument shows that $Y^{1,2}_0(\mathbb{R}^n) = Y^{1,2}(\mathbb{R}^n)$.

We only have to show that the inclusion is strict. To this end, consider

$$f(x) := \frac{1}{(1+|x|)^{n/m+1/2}} \tag{2.3.6}$$

with $\frac{2n}{n-1} < m < 2^*$. A direct computation shows that

$$||Df||_{L^2(\mathbb{R}^n)} < \infty, \quad ||f||_{L^{2^*}(\mathbb{R}^n)} < \infty, \quad \text{and} \quad ||f||_{L^2(\mathbb{R}^n)} = \infty, \tag{2.3.7}$$

so that

$$f \in Y^{1,2}(\mathbb{R}^n) \setminus W^{1,2}(\mathbb{R}^n). \tag{2.3.8}$$

Therefore, $W^{1,2}(\mathbb{R}^n) \subsetneq Y^{1,2}(\mathbb{R}^n)$. $\square$

Lemma 2.3.7. *If $|\Omega| < \infty$, then we have the relations*

$$W_0^{1,2}(\Omega) = Y_0^{1,2}(\Omega) \hookrightarrow Y^{1,2}(\Omega) \hookrightarrow W^{1,2}(\Omega), \tag{2.3.9}$$

where the last inclusion may be an equality for certain domains (see, e.g., the next Lemma), and the norm of the embeddings $Y^{1,2}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ and $Y_0^{1,2}(\Omega) \hookrightarrow W_0^{1,2}(\Omega)$ depends on $|\Omega|$.

Proof. One side of the first equality in (2.3.9) is due to Lemma 2.3.1. On the other hand, for $u \in Y_0^{1,2}(\Omega)$ (or more generally, $u \in Y^{1,2}(\Omega)$), since $|\Omega| < \infty$, we have by the Hölder inequality

$$||u||_{L^2(\Omega)} \leq C_\Omega ||u||_{L^{2^*}(\Omega)}. \tag{2.3.10}$$

Therefore, $Y^{1,2}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ and we can prove that $Y_0^{1,2}(\Omega) \hookrightarrow W_0^{1,2}(\Omega)$ roughly the same way as Lemma 2.3.1, using (2.3.10) to make sure that the sequence which is Cauchy in $Y_0^{1,2}(\Omega)$ is also Cauchy in $W_0^{1,2}(\Omega)$. Together with (2.3.4), this finishes the proof of the lemma. $\square$

Now, the opposite inclusion, $W^{1,2}(\Omega) \hookrightarrow Y^{1,2}(\Omega)$, is a question of validity of the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$. It may fail, but it holds, e.g., for Lipschitz domains. Following [48], we adopt the following definitions.

We say that Ω is a Lipschitz graph domain (or special Lipschitz domain) if there exists a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega = \{(x', x_n) : x_n > \phi(x')\}.$$

We say that Ω is a Lipschitz domain (or a minimally smooth domain, following Stein's terminology) if there exists an $\varepsilon > 0$, $N \in \mathbb{N}$, $M > 0$, and a sequence of open sets $U_1, \dots, U_m, \dots$ along with the corresponding Lipschitz functions $\phi_1, \dots, \phi_m, \dots$ defined on $\mathbb{R}^{n-1}$ and having a Lipschitz constant bounded by M, such that

1. If $x \in \partial\Omega$ then $B(x, \varepsilon) \subset U_i$ for some i.

2. No point of $\mathbb{R}^n$ is contained in more than N of the U_i's.

3. For each i we have, up to rotation, that

$$U_i \cap \Omega = U_i \cap \{(x', x_n) : x_n > \phi_i(x')\}.$$

If Ω satisfies the definition above and is bounded, we refer to it as a bounded Lipschitz domain.

Definition 2.3.8. *We say that Ω is a Sobolev extension domain if there exists a linear mapping $\mathcal{E} : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^n)$ and a constant $C_{\mathcal{E}} > 0$ such that for all $u \in W^{1,2}(\Omega)$,*

$$\mathcal{E}u|_{\Omega} = u \tag{2.3.11}$$

$$||\mathcal{E}u||_{W^{1,2}(\mathbb{R}^n)} \leq C_{\mathcal{E}} \, ||u||_{W^{1,2}(\Omega)} \, . \tag{2.3.12}$$

Theorem 2.3.9 ([48],VI, §3.3)**.** *Lipschitz domains are Sobolev extension domains. The constant of the corresponding extension operator, $C_{\mathcal{E}}$, depends on the number of graphs and their Lipschitz constants.*

Lemma 2.3.10. *If Ω is a Sobolev extension domain, then we have the inclusion (which may be equality)*

$$W^{1,2}(\Omega) \hookrightarrow Y^{1,2}(\Omega) \, , \tag{2.3.13}$$

with the constant in the accompanying estimate for norms depending on $C_{\mathcal{E}}$.

Proof. If Ω is a Sobolev extension domain, then it follows from (2.3.11), (2.3.12), and Lemma 2.3.6 that for all $u \in W^{1,2}(\Omega)$ we have $\mathcal{E}u \in W^{1,2}(\mathbb{R}^n) \hookrightarrow Y^{1,2}(\mathbb{R}^n)$ and

$$\begin{aligned} ||u||_{L^{2^*}(\Omega)} \leq ||\mathcal{E}u||_{L^{2^*}(\mathbb{R}^n)} &\leq C_n(||\mathcal{E}u||_{L^2(\mathbb{R}^n)} + ||D\,(\mathcal{E}u)||_{L^2(\mathbb{R}^n)}) \\ &\leq C_n C_{\mathcal{E}}(||u||_{L^2(\Omega)} + ||Du||_{L^2(\Omega)}). \end{aligned}$$

□

Corollary 2.3.11. *If Ω is a bounded Lipschitz domain, then we have the following relations:*

$$W_0^{1,2}(\Omega) = Y_0^{1,2}(\Omega) \hookrightarrow Y^{1,2}(\Omega) = W^{1,2}(\Omega) \, . \tag{2.3.14}$$

Lemma 2.3.12. *If Ω is a Lipschitz graph domain, then we have the following relations:*

$$W_0^{1,2}(\Omega) \hookrightarrow Y_0^{1,2}(\Omega) \overset{not\ comparable}{\longleftrightarrow} W^{1,2}(\Omega) \hookrightarrow Y^{1,2}(\Omega) \, ,$$

where the inclusions cannot be made equalities.

Proof. The inclusions are given by Lemmas 2.3.1 and 2.3.10.

Without loss of generality, assume $0 \in \partial\Omega$. Let $\Gamma \subset \Omega$ be a cone with its vertex at 0 and its axis in the x_n-direction. Define

$$\gamma := \{x \in \Gamma : \text{dist}(x, \partial\Gamma) > 1\}.$$

Let $\zeta \in C_c^\infty(\Gamma)$ be a smooth cutoff function such that $\zeta \equiv 1$ in γ, $\zeta \equiv 0$ in $\Omega \setminus \Gamma$, and $|D\zeta| \leq C$. Note that $\zeta \equiv 0$ on $\partial\Omega$. Let $f(x)$ be as in the counterexample given by (2.3.6) with $\frac{2n}{n-1} < m < 2^*$. Consider

$$g(x) := \zeta(x) f(x).$$

Then, a computation similar to that which gives (2.3.7) also gives

$$g \in L^{2^*}(\Omega) \setminus L^2(\Omega).$$

It remains only to show that $Dg \in L^2(\Gamma \setminus \gamma)$. Since the cones γ and Γ have equal aperture, we have for sufficiently large s,

$$|(\Gamma \setminus \gamma) \cap \partial B_s(0)| \leq C s^{n-2}.$$

Consequently, a direct computation shows

$$||f||_{L^2(\Gamma\setminus\gamma)} < \infty.$$

Notice that for $t > 1$, $(\Gamma \setminus \gamma) \cap \{x_n = t\}$ forms a $(n-1)$-dimensional annulus of width 1. Thus, we have

$$|(\Gamma \setminus \gamma) \cap \partial B_s(0)| \leq C s^{n-2}, \quad \forall s > 1,$$

and

$$||f||_{L^2(\Gamma\setminus\gamma)} \leq \int_{B_1} |f|^2 + C \int_1^\infty |f(s)|^2 s^{n-2}\, ds \leq C + C \int_1^\infty s^{(1-2/m)n-3}\, ds < \infty,$$

where in the last step we have used that $(1 - 2/m)\, n < 2$.

Therefore,

$$\int_\Omega |Dg|^2 \leq 2 \int_{\Gamma\setminus\gamma} |D\zeta|^2 |f|^2 + 2 \int_\Gamma |\zeta^2| |Df|^2 \leq C \left[||f||_{L^2(\Gamma\setminus\gamma)} + \int_\Omega |Df|^2 \right] < \infty$$

so that $g \in Y^{1,2}(\Omega)$. As in the proof of Lemma 2.3.6, multiplying g by smooth cut-offs ζ_R, we obtain a sequence of $C_c^\infty(\Omega)$ functions that approximate g in the $Y^{1,2}(\Omega)$-norm. Thus, $g \in Y_0^{1,2}(\Omega)$ or more precisely,

$$g \in Y_0^{1,2}(\Omega) \setminus W^{1,2}(\Omega).$$

Therefore, $Y_0^{1,2}(\Omega) \not\subseteq W^{1,2}(\Omega)$. The fact that the opposite inclusion fails is obvious as elements of $W^{1,2}(\Omega)$ do not need to have trace zero on $\partial\Omega$ (in the sense of approximation by smooth compactly supported functions). □

2.3.2 The auxiliary function $m(x, V)$

As can be gleaned from the discussion in the Introduction, in the particular context of the Schrödinger operator it is convenient to work in the weighted Sobolev space, with the weight given by the maximal function $m(x, V)$. In this section, we set the ground by recalling a number of its properties, mainly from [44]. Other versions of these lemmas and definitions appeared in [43] and [45], and are related to the ideas of Fefferman and Phong [18]. We omit the proofs in our exposition.

We say that $V \in B_p$ for some $1 < p < \infty$ if $V \geq 0$ and there exists a constant C so that for any ball $B \subset \mathbb{R}^n$,

$$\left(\fint_B V(x)^p \, dx \right)^{1/p} \leq C \fint_B V(x) \, dx. \tag{2.3.15}$$

If $V \in B_p$, then it follows from an application of the Hölder inequality that $V \in B_q$ for any $q < p$. If $V \in B_p$, then V is a Muckenhoupt A_∞ weight function [49]. Therefore, $V(x)\,dx$ is a doubling measure. That is, there exists a constant C_0 such that

$$\int_{B(x,2r)} V(y) \, dy \leq C_0 \int_{B(x,r)} V(y) \, dy.$$

This fact is very useful in establishing the results below. We now define

$$\psi(x, r; V) = \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy. \tag{2.3.16}$$

We will at times use the shorter notation $\psi(x, r)$ when it is understood that this function is associated to V.

We assume that $V \in B_p$ for some $p \in \left[\frac{n}{2}, \infty\right)$. In fact, it follows from the self-improvement result for reverse Hölder classes that $V \in B_p$ for some $p \in \left(\frac{n}{2}, \infty\right)$ ([21]). Therefore, we will assume throughout that the inequality is strict.

Lemma 2.3.13 (Lemma 1.2, [44]). *If $V \in B_p$, then there exists a constant $C > 0$ so that for any $0 < r < R < \infty$,*

$$\psi(x, r; V) \leq C \left(\frac{r}{R}\right)^{2-\frac{n}{p}} \psi(x, R; V).$$

The proof of Lemma 2.3.13 uses the reverse Hölder inequality (2.3.15) as well the Hölder inequality.

As $V \geq 0$, then for every $x \in \mathbb{R}^n$, either there exists $r > 0$ so that $\psi(x, r; V) > 0$ or $V \equiv 0$ a.e. in $\mathbb{R}^n$. For now, we assume that $V \not\equiv 0$. Since $p > \frac{n}{2}$, the power $2 - \frac{n}{p} > 0$ and

$$\lim_{r \to 0^+} \psi(x, r; V) = 0, \tag{2.3.17}$$

$$\lim_{r \to \infty} \psi(x, r; V) = \infty. \tag{2.3.18}$$

This leads to the following definition.

Definition 2.3.14. *For $x \in \mathbb{R}^n$, the function $m(x, V)$ is defined by*

$$\frac{1}{m(x, V)} = \sup_{r > 0} \{r : \psi(x, r; V) \leq 1\}. \tag{2.3.19}$$

It follows from (2.3.17) and (2.3.18) that $0 < m(x, V) < \infty$ and for every $x \in \mathbb{R}^n$

$$\psi\left(x, \frac{1}{m(x, V)}; V\right) = 1. \tag{2.3.20}$$

Furthermore, from Lemma 2.3.13, if $\psi(x, r; V) \sim 1$, then $r \sim \frac{1}{m(x,V)}$. If $r = \dfrac{1}{m(x, V)}$ then $\fint_{B(x,r)} V(y)\, dy = \dfrac{1}{\omega_n r^2}$, where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

Lemma 2.3.15 (Lemma 1.4, [44]). *There exist constants $C, c, k_0 > 0$ so that for any $x, y \in \mathbb{R}^n$,*

(a) $m(x, V) \sim m(y, V)$ *if* $|x - y| \leq \dfrac{C}{m(x, V)}$,

(b) $m(y,V) \le C\left[1+|x-y|\,m(x,V)\right]^{k_0} m(x,V)$,

(c) $m(y,V) \ge \dfrac{c\,m(x,V)}{\left[1+|x-y|\,m(x,V)\right]^{k_0/(k_0+1)}}$.

Corollary 2.3.16 (Corollary 1.5, [44])**.** *There exist constants $C, c, k_0 > 0$ so that for any $x, y \in \mathbb{R}^n$,*

$$c\left[1+|x-y|\,m(y,V)\right]^{1/(k_0+1)} \le 1+|x-y|\,m(x,V) \le C\left[1+|x-y|\,m(y,V)\right]^{k_0+1}.$$

Remark 2.3.17. Another important consequence of Lemma 2.3.15 is that $m(x,V)$ is locally bounded from above and below. More specifically, for any bounded open set $U \subset \mathbb{R}^n$, there exists a constant $C = C_U > 0$, depending on U and on the constants in Lemma 2.3.15, such that

$$\frac{1}{C} \le m(x,V) \le C, \qquad \text{for any } x \in U.$$

Indeed, the collection $\left\{B_{1/m(x,V)}(x)\right\}_{x\in U}$ is an open covering of $\overline{U}$. Since $\overline{U}$ is compact, then there exists a finite collection of points, $x_1, \ldots, x_M$, such that $\overline{U} \subset \bigcup_{i=1}^{M} B_{1/m(x_i,V)}(x_i)$. It follows from Lemma 2.3.15 that there exists $C > 0$, depending on V, n, so that for any $x \in \overline{U}$, $C^{-1}\min\{m(x_i,V)\}_{i=1}^{M} \le m(x,V) \le C\max\{m(x_i,V)\}_{i=1}^{M}$. In other words, $m(x,V)$ is bounded above and below on $\overline{U}$, and consequently on U.

Lemma 2.3.18 (Lemma 1.8, [44])**.** *There exist constants $C, k_0 > 0$ so that if $R \ge \dfrac{1}{m(x,V)}$*

$$\frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\,dy \le C\left[Rm(x,V)\right]^{k_0}.$$

The last lemma that we will quote from [44] is the Fefferman-Phong inequality.

Lemma 2.3.19 (Lemma 1.9 [44], see also [18])**.** *If $u \in C_c^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |u(x)|^2 m(x,V)^2\,dx \le C\left[\int_{\mathbb{R}^n} |Du(x)|^2\,dx + \int_{\mathbb{R}^n} |u(x)|^2 V(x)\,dx\right].$$

If $V \equiv 0$, then $m(x,V) \equiv 0$ and the previous four results are automatically satisfied.

2.3.3 The weighted Sobolev space $W_V^{1,2}$

Assume $V \in B_p$ for some $p > \frac{n}{2}$. For any open set $\Omega \subset \mathbb{R}^n$, we define the space $W_V^{1,2}(\Omega)$ as the family of all weakly differentiable functions $u \in L^2\left(\Omega, m(x,V)^2\,dx\right)$ whose weak derivatives are functions in $L^2(\Omega, dx)$. The norm and inner product on $W_V^{1,2}(\Omega)$ are given by

$$\begin{aligned} ||u||^2_{W_V^{1,2}(\Omega)} &:= ||u\, m(\cdot, V)||^2_{L^2(\Omega)} + ||Du||^2_{L^2(\Omega)} \\ \langle u, v\rangle_{W_V^{1,2}(\Omega)} &:= \langle um(\cdot, V), vm(\cdot, V)\rangle_{L^2(\Omega)} + \langle Du, Dv\rangle_{L^2(\Omega)}. \end{aligned}$$

$W_{0,V}^{1,2}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W_V^{1,2}(\Omega)$.

We define spaces $\hat{W}_V^{1,2}(\Omega)$ and $\hat{W}_{0,V}^{1,2}(\Omega)$ analogously, but with $V(x)$ in place of $m(x,V)^2$ in the norms. That is, $\hat{W}_V^{1,2}(\Omega)$ is defined to be the family of weakly differentiable functions $u \in L^2(\Omega, V dx)$ whose weak derivatives are functions in $L^2(\Omega, dx)$. The norm and inner product on $\hat{W}_V^{1,2}(\Omega)$ are given by

$$\begin{aligned} ||u||^2_{\hat{W}_V^{1,2}(\Omega)} &:= ||u\, V||_{L^2(\Omega)} + ||Du||^2_{L^2(\Omega)} \\ \langle u, v\rangle_{\hat{W}_V^{1,2}(\Omega)} &:= \left\langle uV^{1/2}, vV^{1/2}\right\rangle_{L^2(\Omega)} + \langle Du, Dv\rangle_{L^2(\Omega)}. \end{aligned}$$

$\hat{W}_{0,V}^{1,2}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $\hat{W}_V^{1,2}(\Omega)$.

Here we prove some essential properties of these spaces and the facts that $W_V^{1,2}(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$ and $\hat{W}_{0,V}^{1,2}(\Omega) = W_{0,V}^{1,2}(\Omega)$.

Remark 2.3.20. Observe that, by Remark 2.3.17, for any bounded open set $U \subset \mathbb{R}^n$, the spaces $W_V^{1,2}(U)$ and $W^{1,2}(U)$ coincide, albeit the norms are only comparable modulo multiplicative constants that depend on U.

First we prove that the weighted Sobolev spaces are indeed Hilbert spaces as defined.

Lemma 2.3.21. *Let $\Omega \subset \mathbb{R}^n$ be open and let $\eta \in L^1_{\text{loc}}(\Omega)$ be real-valued with $\eta > 0$ a.e. The inner product*

$$\langle u, v\rangle_{L^2(\Omega, \eta(x)dx)} = \int_\Omega u(x)\,\overline{v(x)}\eta(x)\,dx,$$

makes $L^2(\Omega, \eta(x)\,dx)$ a Hilbert space.

Proof. It is easy to check that $L^2(\Omega, \eta(x)\,dx)$ is a vector space and $\langle\cdot,\cdot\rangle_{L^2(\Omega,\eta(x)dx)}$ defines an inner product that generates a norm on the space.

To prove completeness, it suffices to show that $L^2(\Omega, \eta(x)\,dx)$ is unitarily equivalent to $L^2(\Omega)$. Consider the map

$$\phi : L^2(\Omega) \to L^2(\Omega, \eta(x)\,dx) : f \mapsto f\eta^{-1/2}.$$

For $f \in L^2(\Omega)$, we have

$$||\phi(f)||_{L^2(\Omega,\eta(x)dx)} = \int_\Omega \left(f\eta^{-1/2}\right)\overline{(f\eta^{-1/2})}\eta = ||f||_{L^2(\Omega)}.$$

Thus, ϕ is injective. For $g \in L^2(\Omega, \eta(x)\,dx)$, take $f = g\eta^{1/2}$. Then $f \in L^2(\Omega)$ and $\phi(f) = g$. Thus, ϕ is surjective. Finally, we check

$$\langle \phi(f), \phi(g)\rangle_{L^2(\Omega,\eta(x)dx)} = \int_\Omega \left(f\eta^{-1/2}\right)\overline{(g\eta^{-1/2})}\eta = \int_\Omega f\overline{g} = \langle f, g\rangle_{L^2(\Omega)}.$$

□

Lemma 2.3.22. *Let $\Omega \subset \mathbb{R}^n$ be open and let $\eta \in L^1_{\mathrm{loc}}(\Omega)$ be real-valued with $\eta > 0$ a.e. Define the space $W^{1,2}_\eta(\Omega)$ as a collection of functions in $L^2(\Omega, \eta(x)dx)$ that are weakly differentiable in Ω with the weak gradient in $L^2(\Omega)$. The inner product*

$$\langle u, v\rangle_{W^{1,2}_\eta(\Omega)} = \int_\Omega Du \cdot \overline{Dv} + \int_\Omega u\,\overline{v}\,\eta$$

makes $W^{1,2}_\eta(\Omega)$ a Hilbert space.

Proof. A quick computation verifies that $\langle\cdot,\cdot\rangle_{W^{1,2}_\eta(\Omega)}$ is an inner product generating the norm on the space. It remains only to show completeness.

Let $\{u_k\}$ be a Cauchy sequence in $W^{1,2}_\eta(\Omega)$. Then $\{u_k\}$ is Cauchy in $L^2(\Omega, \eta(x)\,dx)$, so by Lemma 2.3.21 there exists $u \in L^2(\Omega, \eta(x)\,dx)$ such that

$$u_k \to u \quad \text{in } L^2(\Omega, \eta(x)\,dx). \tag{2.3.21}$$

Furthermore, for $\alpha = 1, \ldots, n$, $\{D_\alpha u_k\}$ is Cauchy in $L^2(\Omega)$, so there exists $v_\alpha \in L^2(\Omega)$ such that

$$D_\alpha u_k \to v_\alpha \quad \text{in } L^2(\Omega). \tag{2.3.22}$$

It remains to show that $v_\alpha = D_\alpha u$. Let $\zeta \in C^\infty_c(\Omega)$. We need to show that

$$\int_\Omega v_\alpha \zeta = -\int_\Omega u D_\alpha \zeta.$$

First, suppose $\operatorname{supp}(\zeta) \subset B$, where B is an open ball that is compactly contained in Ω. The Poincaré inequality yields

$$\begin{aligned}\left|\left|\left(u_k - \fint_B u_k\right) - \left(u_j - \fint_B u_j\right)\right|\right|_{L^2(B)} &= \left|\left|u_k - u_j - \fint_B (u_k - u_j)\right|\right|_{L^2(B)} \\ &\leq C \left|\left|D(u_k - u_j)\right|\right|_{L^2(B)}.\end{aligned}$$

Therefore, $\{u_k - c_k\}$ is Cauchy in $L^2(B)$, with $c_k = \fint_B u_k$. Thus, there exists $\tilde{u} \in L^2(B)$ such that

$$u_k - c_k \to \tilde{u} \quad \text{in } L^2(B). \tag{2.3.23}$$

By Hölder's inequality,

$$\begin{aligned}\left|\left|(u_k - c_k - \tilde{u})\eta^{1/2}\right|\right|_{L^1(B)} &\leq \left|\left|u_k - c_k - \tilde{u}\right|\right|_{L^2(B)} \left|\left|\eta^{1/2}\right|\right|_{L^2(B)} \\ &= \left|\left|u_k - c_k - \tilde{u}\right|\right|_{L^2(B)} \left|\left|\eta\right|\right|_{L^1(B)}^{1/2} \to 0.\end{aligned} \tag{2.3.24}$$

Therefore, by (2.3.24),

$$(u_k - c_k)\eta^{1/2} \to \tilde{u}\eta^{1/2} \quad \text{in } L^1(B). \tag{2.3.25}$$

By (2.3.21), $u_k\eta^{1/2} \to u\eta^{1/2}$ in $L^2(B)$, so it follows that

$$u_k\eta^{1/2} \to u\eta^{1/2} \quad \text{in } L^1(B). \tag{2.3.26}$$

Combining the previous two results shows that

$$c_k\eta^{1/2} \to (u - \tilde{u})\,\eta^{1/2} \quad \text{in } L^1(B).$$

Since each c_k is a constant, it follows that $\lim_{k\to\infty} c_k = c$, where c is some fixed constant. This fact, in combination with (2.3.25), implies that

$$u_k\eta^{1/2} \to (\tilde{u} + c)\,\eta^{1/2} \quad \text{in } L^1(B).$$

With (2.3.26), using that η is almost everywhere non-vanishing, we conclude that $\tilde{u} + c = u$ a.e. in B. From (2.3.23) and the fact that $\{c_k\}$ is a convergent sequence of real numbers, we have

$$u_k \to \tilde{u} + c = u \quad \text{in } L^2(B). \tag{2.3.27}$$

Therefore, by (2.3.22) and (2.3.27),

$$\int_B v_\alpha \zeta = \lim_{k\to\infty} \int_B D_\alpha u_k \zeta = -\lim_{k\to\infty} \int_B u_k D_\alpha \zeta = -\int_B u D_\alpha \zeta. \tag{2.3.28}$$

Now, for any $\zeta \in C_c^\infty(\Omega)$, we can cover $\mathrm{supp}(\zeta)$ with finitely many balls, $\{B_i\}$, with each B_i compactly contained in Ω. Using a partition of unity argument and the result (2.3.28), we obtain the desired equality. □

Corollary 2.3.23. *Let $\Omega \subset \mathbb{R}^n$ be an open set. The spaces $W_V^{1,2}(\Omega)$, $\hat{W}_V^{1,2}(\Omega)$, $W_{0,V}^{1,2}(\Omega)$, and $\hat{W}_{0,V}^{1,2}(\Omega)$ are Hilbert spaces.*

Proof. This follows directly from the previous lemma and the fact that $W_{0,V}^{1,2}(\Omega)$ and $\hat{W}_{0,V}^{1,2}(\Omega)$ are defined as the closure of $C_c^\infty(\Omega)$ in their respective spaces. □

The following lemma shows an important relationship between $W_V^{1,2}(\mathbb{R}^n)$ and $\hat{W}_V^{1,2}(\mathbb{R}^n)$.

Lemma 2.3.24. *Assume that $V \in B_p$ for some $p > \frac{n}{2}$. Then for any $u \in W_V^{1,2}(\mathbb{R}^n)$,*

$$\begin{aligned}\int_{\mathbb{R}^n} V(x)\,|u(x)|^2\,dx &\le C_{V,n}\left(\int_{\mathbb{R}^n} |Du(x)|^2\,dx + \int_{\mathbb{R}^n} |u(x)|^2\, m(x,V)^2\,dx\right)\\ &= C_{V,n}\,||u||^2_{W_V^{1,2}(\mathbb{R}^n)}.\end{aligned} \tag{2.3.29}$$

Conversely, for any $u \in \hat{W}_V^{1,2}(\mathbb{R}^n)$

$$\begin{aligned}\int_{\mathbb{R}^n} |u(x)|^2\, m(x,V)^2\,dx &\le C_{V,n}\left(\int_{\mathbb{R}^n} |Du(x)|^2\,dx + \int_{\mathbb{R}^n} |u(x)|^2\, V(x)\,dx\right)\\ &= C_{V,n}\,||u||^2_{\hat{W}_V^{1,2}(\mathbb{R}^n)}.\end{aligned} \tag{2.3.30}$$

In other words, $W_V^{1,2}(\mathbb{R}^n) = \hat{W}_V^{1,2}(\mathbb{R}^n)$.

Proof. This is essentially Theorem 1.13 in [46]. We only remark that our $V\,dx$ satisfies the conditions of $d\mu$ in the aforementioned theorem by Remark 0.10 in [46], and that the functions with $Du \in L^2(\mathbb{R}^n)$ are $L^2_{loc}(\mathbb{R}^n)$ – this is a standard part of the proof of the Poincaré inequality (see, e.g., [37], 1.1.2). □

If $\Omega \subset \mathbb{R}^n$ is open and connected, then a similar relationship holds for $W_{0,V}^{1,2}(\Omega)$ and $\hat{W}_{0,V}^{1,2}(\Omega)$ and we have the following result.

Lemma 2.3.25. *Assume that $V \in B_p$ for some $p > \frac{n}{2}$. Then for any open set $\Omega \subset \mathbb{R}^n$ we have $W_{0,V}^{1,2}(\Omega) = \hat{W}_{0,V}^{1,2}(\Omega)$, and $\|\cdot\|_{W_{0,V}^{1,2}(\Omega)} \approx \|\cdot\|_{\hat{W}_{0,V}^{1,2}(\Omega)}$ with implicit constants depending on dimension and the B_p constant of V only.*

Proof. Let $u \in W_{0,V}^{1,2}(\Omega)$. By definition, there exists $u_i \in C_c^\infty(\Omega)$ so that $\lim_{i\to\infty} ||u_i - u||_{W_V^{1,2}(\Omega)} = 0$. Applying Lemma 2.3.24 to $u_i - u_k$, we deduce that the sequence $\{u_i\}_{i=1}^\infty$ is Cauchy in $\hat{W}_{0,V}^{1,2}(\Omega)$. Hence, it has a limit in $\hat{W}_{0,V}^{1,2}(\Omega)$ and this limit must coincide with u a.e. since $V > 0$ a.e. and $m(x, V) > 0$ for all $x \in \mathbb{R}^n$. Applying again Lemma 2.3.24, we deduce the desired control of norms. The same argument works in the converse direction. □

2.3.4 Smoothing and approximation

Here we build on Lemma 2.3.5 and collect some results that are related to approximation by smooth functions.

Lemma 2.3.26 (Local approximation by smooth functions)**.** *Let $U \subset \mathbb{R}^n$ be open. Let $\mathbf{F}(U)$ be either $Y^{1,2}(U)$, $W^{1,2}(U)$ or $W_V^{1,2}(U)$. Assume that $u \in \mathbf{F}(U)$, and set $u^\varepsilon = \phi_\varepsilon * u$ in U_ε. Then $u^\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$ and $u^\varepsilon \to u$ in $\mathbf{F}_{\text{loc}}(U)$ as $\varepsilon \to 0$.*

The case of $\mathbf{F}(U) = W^{1,2}(U)$ appears in [17], and the case $\mathbf{F}(U) = W_V^{1,2}(U)$ is the exact same statement due to the local nature of the result and Remark 2.3.20. The case of $\mathbf{F}(U) = Y^{1,2}(U)$ is a slight modification of the aforementioned proof in [17], and we omit it.

Lemma 2.3.27 (Global approximation by smooth functions)**.** *Assume that U is bounded. Let $\mathbf{F}(U)$ be either $Y^{1,2}(U)$, $W^{1,2}(U)$ or $W_V^{1,2}(U)$. If $u \in \mathbf{F}(U)$, then there exists a sequence $\{u_k\}_{k=1}^\infty \subset C^\infty(U) \cap \mathbf{F}(U)$ such that $\lim_{k\to\infty} u_k = u$ in $\mathbf{F}(U)$.*

When $\mathbf{F}(U) = W^{1,2}(U)$, this is Theorem 2 from §5.3.2 of [17], the case $\mathbf{F}(U) = W_V^{1,2}(U)$ is the same due to boundedness of U and Remark 2.3.20, and the case $\mathbf{F}(U) = Y^{1,2}(U)$ is proved in an analogous way. However, we outline the proof here as some elements of it will be useful later.

Proof. We have that $U = \displaystyle\bigcup_{k=1}^\infty U_k$ where $U_k = \left\{x \in U : \text{dist}(x, \partial U) > \frac{1}{k}\right\}$. Set $W_k = U_{k+3} - \overline{U}_{k+1}$. Choose $W_0 \Subset U$ so that $U = \displaystyle\bigcup_{k=0}^\infty W_k$. Let $\{\zeta_k\}_{0=1}^\infty$ be a smooth partition of unity subordinate to $\{W_k\}_{k=1}^{\infty}$. In other words, for each k, $0 \le \zeta_k \le 1$, $\zeta_k \in C_c^{\infty}(W_k)$, and $\sum_{k=1}^{\infty} \zeta_k = 1$ on U. Let $u \in \mathbf{F}(U)$. Since each $\zeta_k \in C_c^{\infty}(U)$, then $\operatorname{supp}(u\zeta_k) \subset W_k$ and by a straightforward argument similar to the proof of Lemma 1(iv) from §5.2 of [17], $u\zeta_k \in \mathbf{F}(U)$.

For each $k = 0, 1, \ldots$, define $X_k = U_{k+4} - \overline{U}_k \supset W_k$. Fix $\delta > 0$. Then, for each k, choose $\varepsilon_k > 0$ so small that $u^k := \phi_{\varepsilon_k} * (u\zeta_k)$ is such that $\operatorname{supp} u^k \subset X_k$ and $\left|\left|u^k - u\zeta_k\right|\right|_{\mathbf{F}(U)} \le \delta 2^{-k-1}$. The second property is guaranteed by the Lemma 2.3.26.

Define $v := \sum_{k=1}^{\infty} u^k$. For any open set $W \Subset U$, there are at most finitely many terms in the sum for v, so it follows that $v \in C^{\infty}(W)$. As $u = \sum_{k=1}^{\infty} u\zeta_k$ then for each $W \Subset U$, we have that

$$||v - u||_{\mathbf{F}(W)} \le \sum_{k=0}^{\infty} \left|\left|u^k - u\zeta_k\right|\right|_{\mathbf{F}(W)} \le \delta \sum_{k=0}^{\infty} 2^{-k-1} = \delta.$$

By taking the supremum over all sets $W \Subset U$, we conclude that $||v - u||_{\mathbf{F}(U)} \le \delta$, and the conclusion of the lemma follows. $\square$

Since the mollification of an a.e. non-negative function is also non-negative, the following corollary is true.

Corollary 2.3.28 (Global approximation by smooth non-negative functions)**.** *Assume that U is bounded. Let $\mathbf{F}(U)$ be either $Y^{1,2}(U)$, $W^{1,2}(U)$ or $W_V^{1,2}(U)$. If $u \in \mathbf{F}(U)$ is non-negative a.e., then there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset C^{\infty}(U) \cap \mathbf{F}(U)$ of non-negative functions such that $\lim_{k\to\infty} u_k = u$ in $\mathbf{F}(U)$.*

Finally, if u is compactly supported in U, then it follows from the previous lemma that u may be approximated by smooth compactly supported functions.

Lemma 2.3.29 (Global approximation by smooth compactly supported functions)**.** *Assume that U is bounded. Let $\mathbf{F}(U)$ be either $Y^{1,2}(U)$, $W^{1,2}(U)$ or $W_V^{1,2}(U)$. If $u \in \mathbf{F}(U)$ and $\operatorname{supp} u \Subset U$, then there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset C_c^{\infty}(U) \cap \mathbf{F}(U)$ such that $\lim_{k\to\infty} u_k = u$ in $\mathbf{F}(U)$.*

We sketch the proof of the lemma.

Proof. Define U_k, W_k, ζ_k as in the proof of Lemma 2.3.27 and conclude as before that each $u\zeta_k \in \mathbf{F}(U)$. Since $\operatorname{supp} u \Subset U$, and $U = \bigcup_{k=0}^{\infty} W_k$, then there exists $M \in \mathcal{N}$ so that $\operatorname{supp} u \subset \bigcup_{k=0}^{M} W_k$. Therefore, $u = \sum_{k=0}^{M} u\zeta_k$. Then (for $k = 0, \ldots, M$) define X_k, u^k as before so that $\operatorname{supp} u^k \subset X_k$ and $\left|\left|u^k - u\zeta_k\right|\right|_{\mathbf{F}(U)} \leq \delta 2^{-k-1}$ and set $v := \sum_{k=1}^{M} u^k$. Since each $u_k \in C_c^{\infty}(U)$, then $v \in C_c^{\infty}(U)$ as well. Moreover,

$$||v - u||_{\mathbf{F}(U)} \leq \sum_{k=0}^{M} \left|\left|u^k - u\zeta_k\right|\right|_{\mathbf{F}(U)} \leq \delta \sum_{k=0}^{M} 2^{-k-1} = \delta$$

and the conclusion follows. □

2.3.5 More general Sobolev spaces

In the last section of this manuscript, we will be using Sobolev spaces with more general integrability, which are defined analogously: $W^{k,p}(\Omega)$ will denote the standard Sobolev space of k-times weakly differentiable functions whose weak derivatives up to order k exist in L^p, $\dot{W}^{k,p}$ is the space of (equivalence classes modulo polynomials) of functions whose weak derivatives of order k belong to L^p, $W_0^{k,p}$ will denote the closure of C_c^{∞} in the $W^{k,p}$ norm. The dual of $W^{k,p}$ will be denoted $W^{-k,p}$, and $\langle f, g\rangle$ will denote the dual pairing of $W^{-k,p}$ with $W^{k,p}$ unless otherwise noted. As is traditional, on the boundary we use L_k^p and $\dot{L}_k^p$ in place of $W^{k,p}$ and $\dot{W}^{k,p}$. All these spaces exhibit properties analogous to those outlined for $p = 2$ in the previous sections. We shall not restate them here as the modifications are straightforward.

It will be important to recall that under certain minimal smoothness assumptions on Ω (in particular, if the domain is Lipschitz), there exists a bounded trace operator acting on Sobolev spaces. We only state this result on $\dot{W}^{1,2}(\mathbb{R}^n_+)$ as it is the only context where it will be used.

There exists a bounded operator

$$\mathrm{Tr}\ : \dot{W}^{1,2}(\mathbb{R}^n_+) \to \dot{L}^2_{1/2}(\mathbb{R}^n),$$

which extends a usual restriction operator on continuous functions, and has a bounded right inverse (cf. [50, Section 2.7.2] for the half-space and [29, Chapter V] for more general domains). Here,

$$\|f\|_{\dot{L}^2_s(\mathbb{R}^n)} := |||\nabla|^s f||_{L^2} = \left(\int |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \quad s \in \mathbb{R}. \tag{2.3.31}$$

We use $\dot{W}^{-k,2}$ or $\dot{L}^2_{-k}$ to indicate the dual of $\dot{W}^{k,2}$ or $\dot{L}^2_k$.

2.3.6 Lorentz and Hardy spaces

We define two alternatives to L^p spaces. Let (X, μ) be a measure space (in our applications either $X = \mathbb{R}^{n-1}$ or $X = \mathbb{R}^n_+$ and $d\mu = dx$ is the Lebesgue measure). For a measurable function, f, define its non-increasing rearrangement by

$$f^*(t) := \inf\{\beta > 0 : m(\beta, f) \leq t\}, \quad t > 0,$$

where

$$m(\beta, f) := \mu(\{x \in X : |f(x)| > \beta\}).$$

Then, for $0 < p < \infty$ and $0 < q \leq \infty$, the Lorentz space $L^{p,q}(X, \mu)$ is defined by the following quasinorm:

$$||f||_{L^{p,q}(X,\mu)} = ||t^{1/p} f^*(t)||_{L^q(\mathbb{R}_+, \frac{dt}{t})}. \tag{2.3.32}$$

In particular, if $q = \infty$ then (2.3.32) becomes

$$||f||_{L^{p,\infty}(X,\mu)} = \left(\sup_{\beta > 0} \beta^p m(\beta, f) \right)^{1/p}. \tag{2.3.33}$$

$L^{p,q}(X, \mu)$ is the set of measurable, complex-valued functions defined on (X, μ) such that the above quasinorm is finite. Lorentz spaces generalize L^p spaces in the sense that $L^{p,p} = L^p$. When X is σ-finite and non-atomic, the following identifications hold:

$$\left(L^{p,q}(X) \right)^* = \begin{cases} \{0\} & \text{if } 0 < p < 1,\ 0 < q \leq \infty, \text{ or } p = 1 \text{ and } 1 < q < \infty, \\ L^\infty(X) & \text{when } p = 1 \text{ and } 1 < q < \infty, \\ L^{p',\infty}(X) & \text{for } 1 < p < \infty \text{ and } 0 < q \leq 1, \\ L^{p',q'}(X) & \text{if } 1 < p, q < \infty. \end{cases} \tag{2.3.34}$$

The other L^p alternative will be the Hardy spaces, H^p. Hardy spaces have a number of equivalent characterizations (as complex analytic functions with boundary values, by

H^p-atoms, etc.). We choose to use the maximal function characterization and we will only need to discuss them on $\mathbb{R}^n$.

Fix $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi \neq 0$, where $\mathcal{S}$ is the Schwarz class of rapidly decaying functions. Define $\psi_\varepsilon(x) := \varepsilon^{-n}\psi(\frac{x}{\varepsilon})$. Define a maximal function on tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ by

$$(Mf)(x) := \sup_{\varepsilon>0} |(f * \psi_\varepsilon)(x)|, \ x \in \mathbb{R}^n.$$

For $0 < p < \infty$, the Hardy spaces on $\mathbb{R}^n$ are then defined

$$\begin{aligned} H^p(\mathbb{R}^n) &:= \{f \in \mathcal{S}'(\mathbb{R}^n) : Mf \in L^p(\mathbb{R}^n)\} \\ ||f||_{H^p(\mathbb{R}^n)} &:= ||Mf||_{L^p(\mathbb{R}^n)}. \end{aligned}$$

$|| \cdot ||_{H^p(\mathbb{R}^n)}$ is a quasinorm for all p and a norm when $p \geq 1$. We get equivalent norms by alternate choices of ψ. H^p is an alternative to L^p in the sense that $H^p = L^p$ for $1 < p < \infty$ and $H^1(\mathbb{R}^n) \subsetneq L^1(\mathbb{R}^n)$. The Hardy spaces are useful in that they extend L^p to $p \leq 1$ in a way that plays nicely, in some sense, with singular integral operators. They also carry useful interpolation properties similar to L^p. See [49, § 2.3.3] for further discussion.

The weak-type Hardy spaces, $H^{p,q}$, are defined similarly by

$$\begin{aligned} H^{p,q}(\mathbb{R}^n) &:= \{f \in \mathcal{S}'(\mathbb{R}^n) : Mf \in L^{p,q}(\mathbb{R}^n)\}, \\ ||f||_{H^{p,q}(\mathbb{R}^n)} &:= ||Mf||_{L^{p,q}(\mathbb{R}^n)}. \end{aligned}$$

Note that $L^p(\mathbb{R}^n) \hookrightarrow L^{p,\infty}(\mathbb{R}^n)$ for all $0 < p < \infty$, and therefore $H^p(\mathbb{R}^n) \hookrightarrow H^{p,\infty}(\mathbb{R}^n)$. Note also that $L^1(\mathbb{R}^n) \hookrightarrow H^{1,\infty}(\mathbb{R}^n)$.

Finally, we will use $\dot{H}^p_1$ and $\dot{H}^{p,q}_1$ for the spaces of tempered distributions modulo constants such that the norms

$$||f||_{\dot{H}^p_1} := ||\nabla f||_{H^p} = ||M\left(\nabla f\right)||_{L^p},$$

$$||f||_{\dot{H}^{p,q}_1} := ||\nabla f||_{H^{p,q}} = ||M\left(\nabla f\right)||_{L^{p,q}}$$

are, respectively, finite.

2.3.7 Real interpolation

Let X_0 and X_1 be two quasinormed vector spaces, continuously embedded into a larger, common topological vector space, X. Such a pair (X_0, X_1) is called a "compatible couple." On the space

$$X_0 + X_1 := \{f \in X : f = f_0 + f_1 \text{ for some } f_0 \in X_0, f_1 \in X_1\},$$

the K-functional at $t > 0$ is defined by

$$K(t, f; X_0, X_1) := \inf\{||f_0||_{X_0} + t||f_1||_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1\},$$

where the infimum is taken over admissible choices of f_0 and f_1. Using this functional, for any $0 < \theta < 1$ and any $0 < q \leq \infty$ we may define a normed space, called an "intermediate space" and denoted $(X_0, X_1)_{\theta,q}$, to be the space of all $f \in X_0 + X_1$ such that the following norm is finite:

$$||f||_{(X_0,X_1)_{\theta,q}} := \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(t, f; X_0, X_1) \right]^q \frac{dt}{t} \right)^{1/q} & \text{if } q < \infty, \\ \sup_{t>0} \left[t^{-\theta} K(t, f; X_0, X_1) \right] & \text{if } q = \infty. \end{cases} \tag{2.3.35}$$

This is one of two common and equivalent methods of defining real interpolation.

For any pair of compatible couples (X_0, X_1) and (Y_0, Y_1) embedded in topological vector spaces X and Y, respectively, we have this important property: If L is any linear operator such that $L : X_0 \to Y_0$ and $L : X_1 \to Y_1$ are bounded, then for any $0 < \theta < 1$ and $0 < q \leq \infty$,

$$L : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$$

is bounded. This is particularly useful when the intermediate spaces characterized by (2.3.35) can be identified (i.e., equal as sets and having equivalent norms) to L^p, or other well-understood, spaces.

There are many results in this direction; we will need two. It is well-known that for $0 < p_0 < p_1 \leq \infty$, the real method of interpolation described above yields

$$(L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta,q} = L^{p,q}(\mathbb{R}^n), \tag{2.3.36}$$

provided $p_0 < q \leq \infty$, $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ (cf., e.g., [10, Theorem 5.2.1]). This implies that under the same conditions,

$$(\dot{H}^{p_0}(\mathbb{R}^n), \dot{H}^{p_1}(\mathbb{R}^n))_{\theta,q} = \dot{H}^{p,q}(\mathbb{R}^n). \tag{2.3.37}$$

Similarly, [19] establishes that for $0 < p_0 < p_1 < \infty$, the real method also gives

$$(H^{p_0,q_0}(\mathbb{R}^n), H^{p_1,q_1}(\mathbb{R}^n))_{\theta,q} = H^{p,q}(\mathbb{R}^n)$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$, and $0 < q \leq \infty$. We will use the particular case when $q_0 = p_0$, $q_1 = p_1$, and $q = \infty$, which gives

$$(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))_{\theta,\infty} = H^{p,\infty}(\mathbb{R}^n). \tag{2.3.38}$$

Chapter 3

Fundamental solutions and Green functions for non-homogeneous systems

3.1 Minimal assumptions on function space and weak definition of elliptic systems

We assume that for any $\Omega \subset \mathbb{R}^n$ open and connected, there exists a Banach space $\mathbf{F}(\Omega)$ consisting of weakly differentiable, vector-valued $L^1_{\text{loc}}(\Omega)$ functions that satisfy the following properties:

A1) Whenever $U \subset \Omega$,

$$\mathbf{u} \in \mathbf{F}(\Omega) \to \mathbf{u}|_U \in \mathbf{F}(U), \quad \text{with } \|\mathbf{u}|_U\|_{\mathbf{F}(U)} \leq \|\mathbf{u}\|_{\mathbf{F}(\Omega)}. \tag{3.1.1}$$

A2) $C_c^\infty(\Omega)^N$ functions belong to $\mathbf{F}(\Omega)$. The space $\mathbf{F}_0(\Omega)$, defined as the closure of $C_c^\infty(\Omega)^N$ with respect to the $\mathbf{F}(\Omega)$-norm, is a Hilbert space with respect to some $\|\cdot\|_{\mathbf{F}_0(\Omega)}$ such that

$$\|\mathbf{u}\|_{\mathbf{F}_0(\Omega)} \approx \|\mathbf{u}\|_{\mathbf{F}(\Omega)} \quad \text{for all } \mathbf{u} \in \mathbf{F}_0(\Omega).$$

A3) The space $\mathbf{F}_0(\Omega)$ is continuously embedded into $Y_0^{1,2}(\Omega)^N$ and respectively, there exists $c_0 > 0$ such that for any $\mathbf{u} \in \mathbf{F}_0(\Omega)$

$$||\mathbf{u}||_{Y_0^{1,2}(\Omega)^N} \le c_0 \, ||\mathbf{u}||_{\mathbf{F}(\Omega)} . \tag{3.1.2}$$

Note that A3) and (2.3.1) imply a homogeneous Sobolev inequality in $\mathbf{F}_0(\Omega)$,

$$||\mathbf{u}||_{L^{2^*}(U)} \lesssim ||D\mathbf{u}||_{L^2(U)} \text{ for any } \mathbf{u} \in \mathbf{F}_0(\Omega), \tag{3.1.3}$$

which will be used repeatedly throughout.

A4) For any $U \subset \mathbb{R}^n$ open and connected

$$\begin{aligned} \mathbf{u} \in \mathbf{F}(\Omega) \text{ and } \xi \in C_c^\infty(U) \quad &\Longrightarrow \quad \mathbf{u}\xi \in \mathbf{F}(\Omega \cap U), \\ \mathbf{u} \in \mathbf{F}(\Omega) \text{ and } \xi \in C_c^\infty(\Omega \cap U) \quad &\Longrightarrow \quad \mathbf{u}\xi \in \mathbf{F}_0(\Omega \cap U), \end{aligned} \tag{3.1.4}$$

with $\|\mathbf{u}\xi\|_{\mathbf{F}(\Omega\cap U)} \le C_\xi \, \|\mathbf{u}\|_{\mathbf{F}(\Omega)}$.

It follows from A4), in particular, that

$$\mathbf{F}(\Omega) \subset Y^{1,2}_{\text{loc}}(\Omega)^N . \tag{3.1.5}$$

Indeed, for any $x \in \Omega$ there exists a ball $B_r(x) \subset \Omega$. If $\mathbf{u} \in \mathbf{F}(\Omega)$ and $\xi \in C_c^\infty(B_r)$, we have $\mathbf{u}\xi \in \mathbf{F}_0(\Omega) \hookrightarrow Y_0^{1,2}(\Omega)^N$. Hence, taking $\xi \equiv 1$ on $B_{r/2}(x)$, we conclude that $\mathbf{u} \in Y^{1,2}\left(B_{r/2}(x)\right)^N$.

Another consequence of (3.1.4) is that for any $U \subset \mathbb{R}^n$ open and connected

$$\text{if } \mathbf{u} \in \mathbf{F}_0(\Omega) \text{ and } \xi \in C_c^\infty(U) \quad \Longrightarrow \quad \mathbf{u}\xi \in \mathbf{F}_0(\Omega \cap U). \tag{3.1.6}$$

Indeed, if $\mathbf{u} \in \mathbf{F}_0(\Omega)$ then there exists a sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)^N$ which converges to $\mathbf{u}$ in $\mathbf{F}(\Omega)$. But then $\{\xi\mathbf{u}_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega\cap U)^N$ is Cauchy in $\mathbf{F}(\Omega\cap U)$ and in $Y^{1,2}(\Omega\cap U)^N$ by (3.1.4) and (3.1.2). Therefore, it converges in $\mathbf{F}(\Omega \cap U)$ and in $Y^{1,2}(\Omega \cap U)^N$ to some element of $\mathbf{F}_0(\Omega \cap U) \hookrightarrow Y_0^{1,2}(\Omega \cap U)^N$, call it $\mathbf{v}$. And it follows that $\mathbf{v} = \mathbf{u}\xi$ as elements of $Y_0^{1,2}(\Omega \cap U)^N$.

For future reference, we mention that for $\Omega, U \subset \mathbb{R}^n$ open and connected, the assumption

$$\mathbf{u} \in \mathbf{F}(\Omega), \quad \mathbf{u} = \mathbf{0} \text{ on } U \cap \partial\Omega, \tag{3.1.7}$$

is always meant in the weak sense of

$$\mathbf{u} \in \mathbf{F}(\Omega) \text{ and } \mathbf{u}\xi \in \mathbf{F}_0(\Omega) \text{ for any } \xi \in C_c^\infty(U). \tag{3.1.8}$$

This definition of (weakly) vanishing on the boundary is independent of the choice of U. Indeed, suppose V is another open and connected subset of $\mathbb{R}^n$ such that $V \cap \partial\Omega = U \cap \partial\Omega$ and let $\xi \in C_c^\infty(V)$. Choose $\psi \in C_c^\infty(U \cap V)$ such that $0 \le \psi \le 1$ and $\psi \equiv 1$ on the support of ξ in some neighborhood of the boundary. Then $\xi(1-\psi)|_\Omega \in C_c^\infty(\Omega)$, so by (3.1.4) we have $\mathbf{u}\xi(1-\psi) \in \mathbf{F}_0(\Omega)$. Additionally, $\xi\psi \in C_c^\infty(U)$, so by (3.1.8), $\mathbf{u}\xi\psi \in \mathbf{F}_0(\Omega)$. Therefore, $\mathbf{u}\xi = \mathbf{u}\xi\psi + \mathbf{u}\xi(1-\psi) \in \mathbf{F}_0(\Omega)$, as desired.

Before stating the remaining properties of $\mathbf{F}(\Omega)$, we define the elliptic operator. Let $\mathbf{A}^{\alpha\beta} = \mathbf{A}^{\alpha\beta}(x)$, $\alpha, \beta \in \{1, \ldots, n\}$, be an $N \times N$ matrix of real bounded measurable coefficients defined on Ω. We assume that $\mathbf{A}^{\alpha\beta}$ satisfies an ellipticity condition in the form of a Gårding inequality,

$$\int_\Omega A_{ij}^{\alpha\beta}(x)\, D_\beta\phi^j(x) D_\alpha\phi^i(x)\, dx \ge \lambda \sum_{i=1}^N \sum_{\alpha=1}^n \int_\Omega \left|D_\alpha\phi^i(x)\right|^2 dx \quad \text{for all } \phi \in \mathbf{F}_0(\Omega) \tag{3.1.9}$$

and a boundedness assumption

$$\sum_{i,j=1}^N \sum_{\alpha,\beta=1}^n \left|A_{ij}^{\alpha\beta}(x)\right|^2 \le \Lambda^2 \quad \text{for all } x \in \Omega, \tag{3.1.10}$$

for some $0 < \lambda, \Lambda < \infty$. Let $\mathbf{V}$ denote the zeroth order term, an $N \times N$ matrix defined on Ω. The first order terms, denoted by $\mathbf{b}^\alpha$ and $\mathbf{d}^\beta$, for each $\alpha, \beta \in \{1, \ldots, n\}$, are $N \times N$ matrices defined on Ω. We assume that there exist $p \in \left(\frac{n}{2}, \infty\right]$ and s, $t \in (n, \infty]$ such that

$$\mathbf{V} \in L_{\text{loc}}^p(\Omega)^{N\times N}, \ \mathbf{b} \in L_{\text{loc}}^s(\Omega)^{n\times N\times N}, \ \mathbf{d} \in L_{\text{loc}}^t(\Omega)^{n\times N\times N}. \tag{3.1.11}$$

We now formally write the operators of interest. In the next paragraph, we will discuss their proper meaning. For every $\mathbf{u} = \left(u^1, \ldots, u^N\right)^T$ in $Y_{\text{loc}}^{1,2}(\Omega)^N$ we define

$$L\mathbf{u} = -D_\alpha\left(\mathbf{A}^{\alpha\beta} D_\beta \mathbf{u}\right). \tag{3.1.12}$$

If we write out (3.1.12) component-wise, we have

$$(L\mathbf{u})^i = -D_\alpha\left(A_{ij}^{\alpha\beta} D_\beta u^j\right), \quad \text{for each } i = 1, \ldots, N.$$

The non-homogeneous second-order operator is written as

$$\begin{aligned}\mathcal{L}\mathbf{u} &:= L\mathbf{u} - D_\alpha\left(\mathbf{b}^\alpha \mathbf{u}\right) + \mathbf{d}^\beta D_\beta \mathbf{u} + \mathbf{V}\mathbf{u} \\ &= -D_\alpha\left(\mathbf{A}^{\alpha\beta} D_\beta \mathbf{u} + \mathbf{b}^\alpha \mathbf{u}\right) + \mathbf{d}^\beta D_\beta \mathbf{u} + \mathbf{V}\mathbf{u},\end{aligned} \tag{3.1.13}$$

or, component-wise,

$$\left(\mathcal{L}\mathbf{u}\right)^i = -D_\alpha\left(A_{ij}^{\alpha\beta} D_\beta u^j + b_{ij}^\alpha u^j\right) + d_{ij}^\beta D_\beta u^j + V_{ij} u^j, \quad \text{for each } i = 1, \ldots, N.$$

The transpose (or adjoint) operator of L, denoted by L^*, is defined by

$$L^* \mathbf{u} = -D_\alpha\left[\left(\mathbf{A}^{\alpha\beta}\right)^* D_\beta \mathbf{u}\right],$$

where $\left(\mathbf{A}^{\alpha\beta}\right)^* = \left(\mathbf{A}^{\beta\alpha}\right)^T$, or equivalently $\left(A_{ij}^{\alpha\beta}\right)^* = A_{ji}^{\beta\alpha}$. Note that the adjoint coefficients, $\left(A_{ij}^{\alpha\beta}\right)^*$ satisfy the same ellipticity assumptions as $A_{ij}^{\alpha\beta}$ given by (3.1.9) and (3.1.10). Take $\left(\mathbf{b}^\alpha\right)^* = \left(\mathbf{d}^\alpha\right)^T$, $\left(\mathbf{d}^\beta\right)^* = \left(\mathbf{b}^\beta\right)^T$, and $\mathbf{V}^* = \mathbf{V}^T$. The adjoint operator to $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L}^* \mathbf{u} &:= L^* \mathbf{u} - D_\alpha\left[\left(\mathbf{b}^\alpha\right)^* \mathbf{u}\right] + \left(\mathbf{d}^\beta\right)^* D_\beta \mathbf{u} + \mathbf{V}^* \mathbf{u} \\ &= -D_\alpha\left[\left(\mathbf{A}^{\beta\alpha}\right)^T D_\beta \mathbf{u} + \left(\mathbf{d}^\alpha\right)^T \mathbf{u}\right] + \left(\mathbf{b}^\beta\right)^T D_\beta \mathbf{u} + \mathbf{V}^T \mathbf{u},\end{aligned} \tag{3.1.14}$$

or

$$\left(\mathcal{L}^* \mathbf{u}\right)^i = -D_\alpha\left(A_{ji}^{\beta\alpha} D_\beta u^j + d_{ji}^\alpha u^j\right) + b_{ji}^\beta D_\beta u^j + V_{ji} u^j, \quad \text{for each } i = 1, \ldots, N.$$

The operators, $L, L^*, \mathcal{L}, \mathcal{L}^*$ are understood in the sense of distributions on Ω. Specifically, for every $\mathbf{u} \in Y_{\text{loc}}^{1,2}\left(\Omega\right)^N$ and $\mathbf{v} \in C_c^\infty\left(\Omega\right)^N$, we use the naturally associated bilinear form and write the action of the functional $\mathcal{L}\mathbf{u}$ on $\mathbf{v}$ as

$$\begin{aligned}\left(\mathcal{L}\mathbf{u}, \mathbf{v}\right) = \mathcal{B}\left[\mathbf{u}, \mathbf{v}\right] &= \int_\Omega \mathbf{A}^{\alpha\beta} D_\beta \mathbf{u} \cdot D_\alpha \mathbf{v} + \mathbf{b}^\alpha\, \mathbf{u} \cdot D_\alpha \mathbf{v} + \mathbf{d}^\beta D_\beta \mathbf{u} \cdot \mathbf{v} + \mathbf{V}\, \mathbf{u} \cdot \mathbf{v} \\ &= \int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha v^i + b_{ij}^\alpha u^j D_\alpha v^i + d_{ij}^\beta D_\beta u^j v^i + V_{ij} u^j v^i.\end{aligned} \tag{3.1.15}$$

It is not hard to check that for such $\mathbf{u}$ and $\mathbf{v}$, and for the coefficients satisfying (3.1.10) and (3.1.11), the bilinear form above is well-defined and finite. Similarly, $\mathcal{B}^*\left[\,\cdot\,,\,\cdot\,\right]$ denotes the bilinear operator associated to $\mathcal{L}^*$, given by

$$\begin{aligned}\left(\mathcal{L}^* \mathbf{u}, \mathbf{v}\right) = \mathcal{B}^*\left[\mathbf{u}, \mathbf{v}\right] &= \int \left(\mathbf{A}^{\beta\alpha}\right)^T D_\beta \mathbf{u} \cdot D_\alpha \mathbf{v} + \left(\mathbf{d}^\alpha\right)^T \mathbf{u} \cdot D_\alpha \mathbf{v} + \left(\mathbf{b}^\beta\right)^T D_\beta \mathbf{u} \cdot \mathbf{v} + \mathbf{V}^T\, \mathbf{u} \cdot \mathbf{v} \\ &= \int A_{ji}^{\beta\alpha} D_\beta u^j D_\alpha v^i + d_{ji}^\alpha u^j D_\alpha v^i + b_{ji}^\beta D_\beta u^j v^i + V_{ji} u^j v^i.\end{aligned} \tag{3.1.16}$$

Clearly,

$$\mathcal{B}[\mathbf{v},\mathbf{u}] = \mathcal{B}^*[\mathbf{u},\mathbf{v}]. \tag{3.1.17}$$

For any vector distribution $\mathbf{f}$ on Ω and $\mathbf{u}$ as above we always understand $\mathcal{L}\mathbf{u} = \mathbf{f}$ on Ω in the weak sense, that is, as $\mathcal{B}[\mathbf{u},\mathbf{v}] = \mathbf{f}(\mathbf{v})$ for all $\mathbf{v} \in C_c^\infty(\Omega)^N$. Typically $\mathbf{f}$ will be an element of some $L^\ell(\Omega)^N$ space and so the action of $\mathbf{f}$ on $\mathbf{v}$ is then simply $\int \mathbf{f}\cdot\mathbf{v}$. The identity $\mathcal{L}^*\mathbf{u} = \mathbf{f}$ is interpreted similarly.

Returning to the properties of the Banach space $\mathbf{F}(\Omega)$ and the associated Hilbert space $\mathbf{F}_0(\Omega)$, we require that $\mathcal{B}$ and $\mathcal{B}^*$ can be extended to bounded and accretive bilinear forms on $\mathbf{F}_0(\Omega) \times \mathbf{F}_0(\Omega)$ so that the Lax-Milgram theorem may be applied in $\mathbf{F}_0(\Omega)$.

A5) *Boundedness hypotheses:*

There exists a constant $\Gamma > 0$ so that for any $\mathbf{u},\mathbf{v} \in \mathbf{F}_0(\Omega)$,

$$\mathcal{B}[\mathbf{u},\mathbf{v}] \le \Gamma\,||\mathbf{u}||_{\mathbf{F}}\,||\mathbf{v}||_{\mathbf{F}}. \tag{3.1.18}$$

A6) *Coercivity hypotheses:*

There exists a constant $\gamma > 0$ so that for any $\mathbf{u} \in \mathbf{F}_0(\Omega)$,

$$\gamma\,||\mathbf{u}||_{\mathbf{F}}^2 \le \mathcal{B}[\mathbf{u},\mathbf{u}] \tag{3.1.19}$$

Finally, we assume

A7) *The Caccioppoli inequality:* If $\mathbf{u} \in \mathbf{F}(\Omega)$ is a weak solution to $\mathcal{L}\mathbf{u} = \mathbf{0}$ in Ω and $\zeta \in C^\infty(\mathbb{R}^n)$ is such that $D\zeta \in C_c^\infty(\Omega)$ and $\zeta\mathbf{u} \in \mathbf{F}_0(\Omega)$, $D_\alpha\zeta\,\mathbf{u} \in L^2(\Omega)^N$, $\alpha = 1,\dots,n$, then

$$\int |D\mathbf{u}|^2\,\zeta^2 \le C\int |\mathbf{u}|^2\,|D\zeta|^2, \tag{3.1.20}$$

where C is a constant that depends on $n, s, t, \gamma, \Gamma, ||\mathbf{b}||_{L^s(\Omega)}$, and $||\mathbf{d}||_{L^t(\Omega)}$. However, C is independent of the set on which ζ and $D\zeta$ are supported.

We remark that the assumption $D\zeta \in C_c^\infty(\Omega)$ implies that ζ is a constant in the exterior of some large ball and, in particular, one can show that under the assumptions of A7) we have also $\zeta^2\mathbf{u} \in \mathbf{F}_0(\Omega)$ (using A4)). This will be useful later on. We also remark that the right-hand side of (3.1.20) is finite by our assumptions.

Finally, let us point out that normally the Caccioppoli inequality will be used either in a ball or in the complement of the ball, that is, $\zeta = \eta$ or $\zeta = 1 - \eta$ for $\eta \in C_c^\infty(B_{2R})$ with $\eta = 1$ on B_R, where B_R is some ball in $\mathbb{R}^n$ possibly intersecting $\partial\Omega$. It is, in fact, only the second case (the complement of the ball) which is needed for construction of the fundamental solution.

Throughout the paper, whenever we assume that A1) – A7) hold, we mean that the assumptions described by A1) – A7) hold for the collections of spaces $\mathbf{F}(\Omega)$ and $\mathbf{F}_0(\Omega)$ and the elliptic operators $\mathcal{L}$ and $\mathcal{L}^*$ with bilinear forms $\mathcal{B}$ and $\mathcal{B}^*$, respectively.

We shall discuss extensively in Section 3.6 and below how the common examples (notably, homogeneous elliptic systems and non-homogeneous elliptic systems with lower order terms in suitable L^p or B_p classes) fit into this framework.

To avoid confusion, we finally point out that $\mathbf{F}(\Omega)$ is of course a *collection* of Banach spaces, indexed by the domain Ω, and the connection between $\mathbf{F}(\Omega_1)$ and $\mathbf{F}(\Omega_2)$ for $\Omega_1 \cap \Omega_2 \neq \emptyset$ is seen through the property A1). That is, $\mathbf{F}(U)$ contains all restrictions of elements of $\mathbf{F}(\Omega)$, when $U \subset \Omega$. We do not assume that any element of $\mathbf{F}(U)$ can be extended to $\mathbf{F}(\Omega)$. This is typical, e.g., for Sobolev spaces $W^{1,2}(\Omega)$, because the extension property might fail on bad domains.

3.2 Fundamental matrices and Green matrices

This section resembles the work done in [27], but we deal here with operators that have lower order terms. In addition to the assumptions regarding $\mathbf{F}(\Omega)$, $\mathbf{F}_0(\Omega)$, $\mathcal{L}$ and $\mathcal{B}$ that are described in the previous section, we assume that all solutions satisfy certain de Giorgi-Nash-Moser estimates. In [27], the authors imposed that all solutions to $L\mathbf{u} = \mathbf{0}$ satisfy bounds on Dirichlet integrals (their results applied only to homogeneous operators). Here, instead, we assume that weak solutions to non-homogeneous equations, $\mathcal{L}\mathbf{u} = \mathbf{f}$, for suitable $\mathbf{f}$, satisfy certain scale-invariant Moser-type estimates and that solutions to homogeneous equations, $\mathcal{L}\mathbf{u} = \mathbf{0}$, are Hölder continuous. We shall make it precise below. To start though, let us introduce a slightly weaker hypothesis (a Moser-type local bound):

- For any $y \in \Omega$, there exists an $R_y \in (0, \infty]$ such that whenever $0 < 2r < R_y$, $\mathbf{f} \in L^\ell(\Omega_r(y))^N$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$, $\mathbf{u} \in \mathbf{F}(\Omega_{2r}(y))$ satisfies $\mathbf{u} = \mathbf{0}$ on $\partial\Omega \cap B_{2r}(y)$

in the weak sense of (3.1.8), and either $\mathcal{L}\mathbf{u} = \mathbf{f}$ or $\mathcal{L}^*\mathbf{u} = \mathbf{f}$ in $\Omega_r(y)$ in the weak sense, then for any $q > 0$ there is a $C > 0$ so that

$$||\mathbf{u}||_{L^\infty(\Omega_{r/2}(y))} \leq C \left[r^{-\frac{n}{q}} ||\mathbf{u}||_{L^q(\Omega_r(y))} + r^{2-\frac{n}{\ell}} ||\mathbf{f}||_{L^\ell(\Omega_r(y))} \right]. \tag{3.2.1}$$

Without loss of generality, we can assume that the righthand side of (3.2.1) is finite. Indeed, if we take $\zeta \in C_c^\infty(B_{2r}(y))$ such that $\zeta \equiv 1$ on $B_r(y)$ then $\mathbf{u}\zeta \in \mathbf{F}_0(\Omega_{2r})$ assures that $\mathbf{u} \in L^{2^*}(\Omega_r)^N$ by the homogenous Sobolev inequality, (3.1.3). Then (3.2.1) shows that $\mathbf{u} \in L^q(\Omega_{r/2})^N$ for any $q < \infty$. Strictly speaking, it would be more coherent then to write (3.2.1) for $r < R_y/4$ but we ignore this minor inconsistency as clearly in practice one can always adjust the constants when proving (3.2.1). If $\ell = \infty$, then we interpret $\frac{1}{\ell}$ to equal 0. This convention will be used throughout.

Note that the constant C in the estimate above is allowed to depend on the choice of $\mathcal{L}$, but it should be independent of r and R_y. In other words, we assume that all solutions satisfy a local scale-invariant Moser boundedness condition.

In this respect, we would like to make the following remark. All boundedness and Hölder continuity conditions on solutions that we impose are local in nature. However, slightly abusing the terminology, we refer to a given condition as local if it only holds for balls of the radius smaller than R_0, for some fixed $R_0 > 0$, depending or not depending on the center of the ball. As such, (3.2.1) is local. Later on, we will also talk about interior estimates which hold for balls inside Ω and boundary estimates in which balls are allowed to intersect the boundary. Either can be local or global depending on whether the size of the balls is restricted, and the interior estimates are of course always local if $\Omega \neq \mathbb{R}^n$. In any case, we are always careful to specify the exact condition.

Remark 3.2.1. If $R_y = \operatorname{dist}(y, \partial\Omega)$ then $\partial\Omega \cap B_r = \emptyset$, hence, in that case, (3.2.1) is merely an interior (rather than a boundary) condition.

3.2.1 A general construction method

First, we establish a supporting lemma that will make the proofs in the following sections more concise. We follow closely the argument in [27].

Lemma 3.2.2. *Let Ω be an open connected subset of $\mathbb{R}^n$. Assume that* A1) – A7) *hold. Then for all $y \in \Omega$, $0 < \rho < d_y := \operatorname{dist}(y, \partial\Omega)$, $k \in \{1, \ldots, N\}$, there exists*

$\mathbf{v}_\rho = \mathbf{v}_{\rho;y,k} \in \mathbf{F}_0(\Omega)$ *such that*

$$\mathcal{B}[\mathbf{v}_\rho, \mathbf{u}] = \fint_{B_\rho(y)} u^k = \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)} u^k, \quad \forall\, \mathbf{u} \in \mathbf{F}_0(\Omega). \tag{3.2.2}$$

If, in addition, (3.2.1) *holds, then there exists a function* $\mathbf{v} = \mathbf{v}_{y,k}$ *and a subsequence* $\{\rho_\mu\}_{\mu=1}^\infty$, $\rho_\mu \to 0$, *such that*

$$\mathbf{v}_{\rho_\mu} \rightharpoonup \mathbf{v} \quad \text{in } W^{1,q}(\Omega_r(y))^N \quad \forall r < \tfrac{1}{2}R_y, \quad \forall q \in \left(1, \frac{n}{n-1}\right), \tag{3.2.3}$$

$$\mathbf{v}_{\rho_\mu} \rightharpoonup \mathbf{v} \quad \text{in } L^{q}(\Omega_r(y))^N \quad \forall r < \tfrac{1}{2}R_y, \quad \forall q \in \left(1, \frac{n}{n-2}\right), \tag{3.2.4}$$

$$\mathbf{v}_{\rho_\mu} \rightharpoonup \mathbf{v} \quad \text{in } Y^{1,2}(\Omega \setminus \Omega_r(y))^N, \quad \forall r > 0. \tag{3.2.5}$$

For any $\phi \in C_c^\infty(\Omega)^N$,

$$\mathcal{B}[\mathbf{v}, \phi] = \phi^k(y). \tag{3.2.6}$$

If $\mathbf{f} \in L_c^\infty(\Omega)^N$ *and* $\mathbf{u} \in \mathbf{F}_0(\Omega)$ *is the unique weak solution to* $\mathcal{L}^*\mathbf{u} = \mathbf{f}$, *then for a.e.* $y \in \Omega$,

$$u^k(y) = \int_\Omega \mathbf{v}\cdot\mathbf{f}. \tag{3.2.7}$$

Furthermore, $\mathbf{v}$ *satisfies the following estimates:*

$$||\mathbf{v}||_{L^{2^*}(\Omega\setminus\Omega_r(y))} \le C r^{1-\frac{n}{2}}, \quad \forall r < \tfrac{1}{2}R_y, \tag{3.2.8}$$

$$||D\mathbf{v}||_{L^{2}(\Omega\setminus\Omega_r(y))} \le C r^{1-\frac{n}{2}}, \quad \forall r < \tfrac{1}{2}R_y, \tag{3.2.9}$$

$$||\mathbf{v}||_{L^{q}(\Omega_r(y))} \le C_q r^{2-n+\frac{n}{q}}, \quad \forall r < \tfrac{1}{2}R_y, \quad \forall q \in \left[1, \tfrac{n}{n-2}\right), \tag{3.2.10}$$

$$||D\mathbf{v}||_{L^{q}(\Omega_r(y))} \le C_q r^{1-n+\frac{n}{q}}, \quad \forall r < \tfrac{1}{2}R_y, \quad \forall q \in \left[1, \tfrac{n}{n-1}\right), \tag{3.2.11}$$

$$|\{x \in \Omega : |\mathbf{v}(x)| > \tau\}| \le C\tau^{-\frac{n}{n-2}}, \quad \forall \tau > \left(\tfrac{1}{2}R_y\right)^{2-n}, \tag{3.2.12}$$

$$|\{x \in \Omega : |D\mathbf{v}(x)| > \tau\}| \le C\tau^{-\frac{n}{n-1}}, \quad \forall \tau > \left(\tfrac{1}{2}R_y\right)^{1-n}, \tag{3.2.13}$$

$$|\mathbf{v}(x)| \le C R_{x,y}^{2-n} \quad \text{for a.e. } x \in \Omega, \text{ where } R_{x,y} := \min\{R_x, R_y, |x-y|\}, \tag{3.2.14}$$

where each constant depends on n, N, c_0, Γ, γ, *and the constants from* (3.1.20) *and* (3.2.1), *and each* C_q *depends additionally on* q.

Proof of Lemma 3.2.2. Let $\mathbf{u} \in \mathbf{F}_0(\Omega)$. Fix $y \in \Omega$, $0 < \rho < d_y$, and $k \in \{1, \dots, N\}$, and consider the linear functional

$$\mathbf{u} \mapsto \fint_{B_\rho(y)} u^k.$$

By the Hölder inequality, (3.1.3), and (3.1.2),

$$\begin{aligned}\left|\fint_{B_\rho(y)} u^k\right| &\le \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)} |\mathbf{u}| \le |B_\rho(y)|^{\frac{2-n}{2n}}\left(\int_\Omega |\mathbf{u}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \\ &\le c_n |B_\rho(y)|^{\frac{2-n}{2n}}\left(\int_\Omega |D\mathbf{u}|^2\right)^{\frac{1}{2}} \\ &\le c_0 c_n \rho^{\frac{2-n}{2}} \|\mathbf{u}\|_{\mathbf{F}(\Omega)}. \end{aligned} \tag{3.2.15}$$

Therefore, the functional is bounded on $\mathbf{F}_0(\Omega)$, and by the Lax-Milgram theorem there exists a unique $\mathbf{v}_\rho \in \mathbf{F}_0(\Omega)$ satisfying (3.2.2). By coercivity of $\mathcal{B}$ given by (3.1.19) along with (3.2.15), we obtain,

$$\gamma \|\mathbf{v}_\rho\|_{\mathbf{F}(\Omega)}^2 \le \mathcal{B}[\mathbf{v}_\rho, \mathbf{v}_\rho] = \left|\fint_{B_\rho(y)} v_\rho^k\right| \le c_0 c_n \rho^{\frac{2-n}{2}} \|\mathbf{v}_\rho\|_{\mathbf{F}(\Omega)}$$

so that

$$\|D\mathbf{v}_\rho\|_{L^2(\Omega)} \le c_0 \|\mathbf{v}_\rho\|_{\mathbf{F}(\Omega)} \le C\rho^{\frac{2-n}{2}}, \tag{3.2.16}$$

where the first inequality is by (3.1.2).

For $\mathbf{f} \in L_c^\infty(\Omega)^N$, consider the linear functional

$$\mathbf{F}_0(\Omega) \ni \mathbf{w} \mapsto \int_\Omega \mathbf{f}\cdot\mathbf{w}.$$

This functional is bounded on $\mathbf{F}_0(\Omega)$ since for every $\mathbf{w} \in \mathbf{F}_0(\Omega)$, and any $\ell \in \left(\frac{n}{2}, \infty\right]$,

$$\left|\int_\Omega \mathbf{f}\cdot\mathbf{w}\right| \le \|\mathbf{f}\|_{L^\ell(\Omega)} \|\mathbf{w}\|_{L^{\frac{2n}{n-2}}(\Omega)} |\operatorname{supp}\mathbf{f}|^{\frac{n+2}{2n}-\frac{1}{\ell}} \le C\|\mathbf{f}\|_{L^\ell(\Omega)} |\operatorname{supp}\mathbf{f}|^{\frac{n+2}{2n}-\frac{1}{\ell}} \|\mathbf{w}\|_{\mathbf{F}(\Omega)}, \tag{3.2.17}$$

where we have again used (3.1.3) and (3.1.2). Then, once again by Lax-Milgram, we obtain $\mathbf{u} \in \mathbf{F}_0(\Omega)$ such that

$$\mathcal{B}^*[\mathbf{u}, \mathbf{w}] = \int_\Omega \mathbf{f}\cdot\mathbf{w}, \quad \forall\, \mathbf{w} \in \mathbf{F}_0(\Omega). \tag{3.2.18}$$

Set $\mathbf{w} = \mathbf{u}$ in (3.2.18) and use the coercivity assumption, (3.1.19), for $\mathcal{B}^*$ and (3.2.17) to get

$$\|\mathbf{u}\|_{\mathbf{F}(\Omega)} \le C\|\mathbf{f}\|_{L^\ell(\Omega)} |\operatorname{supp}\mathbf{f}|^{\frac{n+2}{2n}-\frac{1}{\ell}}. \tag{3.2.19}$$

Also, if we take $\mathbf{w} = \mathbf{v}_\rho$ in (3.2.18), we get

$$\int_\Omega \mathbf{f}\cdot\mathbf{v}_\rho = \mathcal{B}^*[\mathbf{u}, \mathbf{v}_\rho] = \mathcal{B}[\mathbf{v}_\rho, \mathbf{u}] = \fint_{B_\rho(y)} u^k. \tag{3.2.20}$$

Let $\mathbf{f} \in L_c^{\infty}(\Omega)^N$ be supported in $\Omega_r(y)$, where $0 < 2r < R_y$, and let $\mathbf{u}$ be as in (3.2.18). Since $\mathbf{u} \in \mathbf{F}_0(\Omega)$, then A1) implies that $\mathbf{u} \in \mathbf{F}(\Omega_{2r})$ and A4) gives $\mathbf{u} = \mathbf{0}$ on $\partial\Omega \cap \Omega_{2r}$ so that (3.2.1) is applicable. Then, by (3.2.1) with $q = \frac{2n}{n-2}$ and $\ell \in \left(\frac{n}{2}, \infty\right]$,

$$||\mathbf{u}||^2_{L^\infty\left(\Omega_{r/2}(y)\right)} \le C\left(r^{2-n}\,||\mathbf{u}||^2_{L^{\frac{2n}{n-2}}(\Omega_r(y))} + r^{4-\frac{2n}{\ell}}\,||\mathbf{f}||^2_{L^\ell(\Omega_r(y))}\right).$$

By (3.1.3), (3.1.2), and (3.2.19) with $\operatorname{supp}\mathbf{f} \subset \Omega_r(y)$,

$$\begin{aligned} ||\mathbf{u}||^2_{L^{2^*}(\Omega_r(y))} \le ||\mathbf{u}||^2_{L^{2^*}(\Omega)} \le C\,||\mathbf{u}||^2_{\mathbf{F}(\Omega)} &\le C\,|\Omega_r(y)|^{1+\frac{2}{n}-\frac{2}{\ell}}\,||\mathbf{f}||^2_{L^\ell(\Omega)} \\ &\le C\,|B_r(y)|^{1+\frac{2}{n}-\frac{2}{\ell}}\,||\mathbf{f}||^2_{L^\ell(\Omega)}, \end{aligned}$$

where, as before, $2^* = \frac{2n}{n-2}$. Combining the previous two inequalities, we get

$$||\mathbf{u}||^2_{L^\infty\left(\Omega_{r/2}(y)\right)} \le C r^{4-\frac{2n}{\ell}}\,||\mathbf{f}||^2_{L^\ell(\Omega)}.$$

Therefore,

$$||\mathbf{u}||_{L^\infty\left(\Omega_{r/2}(y)\right)} \le C r^{2-\frac{n}{\ell}}\,||\mathbf{f}||_{L^\ell(\Omega)} = C r^{2-\frac{n}{\ell}}\,||\mathbf{f}||_{L^\ell(\Omega_r(y))}, \quad \forall \ell \in \left(\frac{n}{2}, \infty\right]. \tag{3.2.21}$$

By (3.2.20) and (3.2.21), if $\rho \le r/2$, $\rho < d_y$, we have

$$\begin{aligned} \left|\int_{\Omega_r(y)} \mathbf{f}\cdot\mathbf{v}_\rho\right| = \left|\int_\Omega \mathbf{f}\cdot\mathbf{v}_\rho\right| &\le \fint_{B_\rho(y)} |\mathbf{u}| \le ||\mathbf{u}||_{L^\infty(B_\rho(y))} \\ &\le ||\mathbf{u}||_{L^\infty(\Omega_{r/2}(y))} \le C r^{2-\frac{n}{\ell}}\,||\mathbf{f}||_{L^\ell(\Omega_r(y))}, \quad \forall \ell \in \left(\frac{n}{2}, \infty\right]. \end{aligned}$$

By duality, this implies that for $r < \frac{1}{2}R_y$,

$$||\mathbf{v}_\rho||_{L^q(\Omega_r(y))} \le C r^{2-n+\frac{n}{q}}, \qquad \text{for all } \rho \le \frac{r}{2}, \rho < d_y, \quad \forall q \in \left[1, \tfrac{n}{n-2}\right). \tag{3.2.22}$$

Fix $x \neq y$ such that $r := \frac{4}{3}|x-y| < \frac{1}{2}R_y$. For $\rho \le r/2, \rho < d_y$, $\mathbf{v}_\rho$ is a weak solution to $\mathcal{L}\mathbf{v}_\rho = \mathbf{0}$ in $\Omega_{r/4}(x)$. Moreover, since $\mathbf{v}_\rho \in \mathbf{F}_0(\Omega)$, then A1) implies that $\mathbf{v}_\rho \in \mathbf{F}\left(\Omega_{r/2}(x)\right)$ and A4) implies that $\mathbf{v}_\rho = \mathbf{0}$ on $\partial\Omega \cap \Omega_{r/2}(x)$ so we may use (3.2.1). Thus, applying (3.2.1) with $q = 1$ and (3.2.22), we get for a.e. $x \in \Omega$ as above,

$$|\mathbf{v}_\rho(x)| \le C r^{-n}\,||\mathbf{v}_\rho||_{L^1\left(\Omega_{r/4}(x)\right)} \le C r^{-n}\,||\mathbf{v}_\rho||_{L^1(\Omega_r(y))} \le C r^{2-n} \approx |x-y|^{2-n}. \tag{3.2.23}$$

Now, for any $r < \frac{1}{2}R_y$ and $\rho \le r/2$, $\rho < d_y$, let ζ be a cut-off function such that

$$\zeta \in C^\infty(\mathbb{R}^n), \quad 0 \le \zeta \le 1, \quad \zeta \equiv 1 \text{ outside } B_r(y), \quad \zeta \equiv 0 \text{ in } B_{r/2}(y), \quad \text{and } |D\zeta| \le C/r. \tag{3.2.24}$$

Then $\mathbf{v}_\rho\zeta, \mathbf{v}_\rho D_\alpha\zeta \in \mathbf{F}_0\left(\Omega \setminus \overline{\Omega_{r/2}(y)}\right)$, for all $\alpha = 1, ..., n$. For the functions $\mathbf{v}_\rho D_\alpha\zeta$, this fact follows from (3.1.6). The function $\mathbf{v}_\rho\zeta$ is a little more delicate since ζ is not compactly supported. However, since ζ equals 1 in the complement of $B_r(y)$, then $1-\zeta$ is compactly supported. Thus, if $\{\mathbf{v}_n\} \subset C_c^\infty(\Omega)^N$ converges to $\mathbf{v}_\rho$ in the $\mathbf{F}(\Omega)$-norm, then, by (3.1.6), $\{\mathbf{v}_n(1-\zeta)\} \subset C_c^\infty(\Omega)^N$ converges to $\mathbf{v}_\rho(1-\zeta)$ in the $\mathbf{F}(\Omega)$-norm . Adding up these statements, we conclude that $\{\mathbf{v}_n\zeta\} \subset C_c^\infty(\Omega)^N$ approximates $\mathbf{v}_\rho\zeta$ in $\mathbf{F}(\Omega)$, as required.

Now, since $\mathcal{L}\mathbf{v}_\rho = \mathbf{0}$ in $\Omega \setminus \overline{\Omega_{r/2}(y)}$, the Caccioppoli inequality, (3.1.20), implies that

$$\int_\Omega \zeta^2 |D\mathbf{v}_\rho|^2 \le C \int_\Omega |D\zeta|^2 |\mathbf{v}_\rho|^2 \le Cr^{-2} \int_{\Omega_r(y)\setminus\Omega_{r/2}(y)} |\mathbf{v}_\rho|^2, \quad \forall \rho \le \frac{r}{2},\, \rho < d_y. \tag{3.2.25}$$

Combining (3.2.25) and (3.2.23), we have for all $r < \frac{1}{2}R_y$ and ζ as above,

$$\begin{aligned} \int_\Omega |D(\zeta\mathbf{v}_\rho)|^2 &\le 2\int_\Omega \zeta^2 |D\mathbf{v}_\rho|^2 + 2\int_\Omega |D\zeta|^2 |\mathbf{v}_\rho|^2 \\ &\le Cr^{-2} \int_{\Omega_r(y)\setminus\Omega_{r/2}(y)} |\mathbf{v}_\rho|^2 \le Cr^{2-n}, \quad \forall \rho \le \frac{r}{2},\, \rho < d_y. \end{aligned} \tag{3.2.26}$$

It follows from (3.1.3) and (3.2.26) that for $r < \frac{1}{2}R_y$,

$$\begin{aligned} \int_{\Omega\setminus\Omega_r(y)} |\mathbf{v}_\rho|^{\frac{2n}{n-2}} &\le \int_\Omega |\zeta\mathbf{v}_\rho|^{\frac{2n}{n-2}} \\ &\le \left(\int_\Omega |D(\zeta\mathbf{v}_\rho)|^2\right)^{\frac{n}{n-2}} \le Cr^{-n}, \quad \forall \rho \le \frac{r}{2},\, \rho < d_y. \end{aligned} \tag{3.2.27}$$

On the other hand, if $\frac{r}{2} < \rho < d_y$, then (3.1.3) and (3.2.16) imply

$$\int_{\Omega\setminus\Omega_r(y)} |\mathbf{v}_\rho|^{\frac{2n}{n-2}} \le \int_\Omega |\mathbf{v}_\rho|^{\frac{2n}{n-2}} \le C\left(\int_\Omega |D\mathbf{v}_\rho|^2\right)^{\frac{n}{n-2}} \le Cr^{-n}. \tag{3.2.28}$$

Therefore, combining the previous two results, we have

$$\int_{\Omega\setminus\Omega_r(y)} |\mathbf{v}_\rho|^{\frac{2n}{n-2}} \le Cr^{-n}, \quad \forall r < \tfrac{1}{2}R_y, \quad \forall\, 0 < \rho < d_y. \tag{3.2.29}$$

Fix $\tau > (R_y/2)^{2-n}$. If $R_y = \infty$, then fix $\tau > 0$. Let $A_\tau = \{x \in \Omega : |\mathbf{v}_\rho| > \tau\}$ and set $r = \tau^{\frac{1}{2-n}}$. Note that $r < \frac{1}{2}R_y$. Then, using (3.2.29), we see that if $0 < \rho < d_y$,

$$|A_\tau \setminus \Omega_r(y)| \le \tau^{-\frac{2n}{n-2}} \int_{A_\tau\setminus\Omega_r(y)} |\mathbf{v}_\rho|^{\frac{2n}{n-2}} = C\tau^{-\frac{n}{n-2}}.$$

Since $|A_\tau \cap \Omega_r(y)| \le |\Omega_r(y)| \le Cr^n = C\tau^{-\frac{n}{n-2}}$, we have

$$|\{x \in \Omega : |\mathbf{v}_\rho(x)| > \tau\}| \le C\tau^{-\frac{n}{n-2}} \quad \text{if } \tau > \left(\frac{R_y}{2}\right)^{2-n}, \quad \forall\, 0 < \rho < d_y. \tag{3.2.30}$$

Fix $r < \frac{1}{2}R_y$ and let ζ be as in (3.2.24). Then (3.2.26) gives

$$\int_{\Omega\setminus\Omega_r(y)} |D\mathbf{v}_\rho|^2 \le Cr^{2-n}, \quad \forall r < \tfrac{1}{2}R_y, \quad \forall \rho \le \frac{r}{2}.$$

Now, if $\frac{r}{2} < \rho < d_y$, we have from (3.2.16) that

$$\int_{\Omega\setminus\Omega_r(y)} |D\mathbf{v}_\rho|^2 \le \int_\Omega |D\mathbf{v}_\rho|^2 \le C\rho^{2-n} \le Cr^{2-n}.$$

Combining the previous two results yields

$$\int_{\Omega\setminus\Omega_r(y)} |D\mathbf{v}_\rho|^2 \le Cr^{2-n}, \quad \forall r < \tfrac{1}{2}R_y, \quad \forall\, 0 < \rho < d_y. \tag{3.2.31}$$

Fix $\tau > (R_y/2)^{1-n}$. If $R_y = \infty$, let $\tau > 0$. Let $A_\tau = \{x \in \Omega : |D\mathbf{v}_\rho| > \tau\}$ and set $r = \tau^{\frac{1}{1-n}}$. Note that $r < \frac{1}{2}R_y$. Then, using (3.2.31), we see that if $0 < \rho < d_y$,

$$|A_\tau \setminus \Omega_r(y)| \le \tau^{-2} \int_{A_\tau\setminus\Omega_r(y)} |D\mathbf{v}_\rho|^2 \le C\tau^{-\frac{n}{n-1}}.$$

Since $|A_\tau \cap \Omega_r(y)| \le Cr^n = C\tau^{-\frac{n}{n-1}}$, then

$$|\{x \in \Omega : |D\mathbf{v}_\rho(x)| > \tau\}| \le C\tau^{-\frac{n}{n-1}} \quad \text{if } \tau > \left(\tfrac{1}{2}R_y\right)^{1-n}, \quad \forall\, 0 < \rho < d_y. \tag{3.2.32}$$

For any $\sigma > (R_y/2)^{1-n}$ and $q > 0$, we have

$$\int_{\Omega_r(y)} |D\mathbf{v}_\rho|^q \le \sigma^q\, |\Omega_r(y)| + \int_{\{|D\mathbf{v}_\rho| > \sigma\}} |D\mathbf{v}_\rho|^q\,.$$

By (3.2.32), for $q \in \left(0, \frac{n}{n-1}\right)$ and $\rho \in (0, d_y)$,

$$\begin{aligned}\int_{\{|D\mathbf{v}_\rho|>\sigma\}} |D\mathbf{v}_\rho|^q &= \int_0^\infty q\tau^{q-1}\, |\{|D\mathbf{v}_\rho| > \max\{\tau,\sigma\}\}|\, d\tau \\ &\le C\sigma^{-\frac{n}{n-1}} \int_0^\sigma q\tau^{q-1}\, d\tau + C\int_\sigma^\infty q\tau^{q-1-\frac{n}{n-1}}\, d\tau = C\left(1 - \frac{q}{q - \frac{n}{n-1}}\right)\sigma^{q-\frac{n}{n-1}}.\end{aligned}$$

Therefore, taking $\sigma = r^{1-n}$, we conclude that

$$\int_{\Omega_r(y)} |D\mathbf{v}_\rho|^q \leq C_q r^{q(1-n)+n}, \quad \forall r < \tfrac{1}{2}R_y, \quad \forall\, 0 < \rho < d_y, \quad \forall q \in \left(0, \tfrac{n}{n-1}\right). \tag{3.2.33}$$

By the same process with (3.2.30) in place of (3.2.32) and $\sigma = r^{2-n}$, we have

$$\int_{\Omega_r(y)} |\mathbf{v}_\rho|^q \leq C_q r^{q(2-n)+n}, \quad \forall r < \tfrac{1}{2}R_y, \quad \forall\, 0 < \rho < d_y, \quad \forall q \in \left(0, \tfrac{n}{n-2}\right). \tag{3.2.34}$$

Fix $q \in \left(1, \frac{n}{n-1}\right)$ and $\tilde{q} \in \left(1, \frac{n}{n-2}\right)$. From (3.2.33) and (3.2.34), it follows that for any $r < \frac{1}{2}R_y$

$$||\mathbf{v}_\rho||_{W^{1,q}(\Omega_r(y))} \leq C(r) \text{ and } ||\mathbf{v}_\rho||_{L^{\tilde{q}}(\Omega_r(y))} \leq C(r) \quad \text{uniformly in } \rho. \tag{3.2.35}$$

Therefore, (using diagonalization) we can show that there exists a sequence $\{\rho_\mu\}_{\mu=1}^{\infty}$ tending to 0 and a function $\mathbf{v} = \mathbf{v}_{y,k}$ such that

$$\mathbf{v}_{\rho_\mu} \rightharpoonup \mathbf{v} \quad \text{in } W^{1,q}(\Omega_r(y))^N \text{ and in } L^{\tilde{q}}(\Omega_r(y))^N, \text{ for all } r < \frac{1}{2}R_y. \tag{3.2.36}$$

Furthermore, for fixed $r_0 < r$, (3.2.29) and (3.2.31) imply uniform bounds on $\mathbf{v}_{\rho_\mu}$ in $Y^{1,2}(\Omega \setminus \Omega_{r_0}(y))^N$ for small ρ_μ. Thus, there exists a subsequence of $\{\rho_\mu\}$ (which we will not rename) and a function $\widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}_{y,k}$ such that

$$\mathbf{v}_{\rho_\mu} \rightharpoonup \widetilde{\mathbf{v}} \quad \text{in } Y^{1,2}(\Omega \setminus \Omega_{r_0}(y))^N. \tag{3.2.37}$$

Since $\mathbf{v} \equiv \widetilde{\mathbf{v}}$ on $\Omega_r(y) \setminus \Omega_{r_0}(y)$, we can extend $\mathbf{v}$ to the entire Ω by setting $\mathbf{v} = \widetilde{\mathbf{v}}$ on $\Omega \setminus \Omega_r(y)$. For ease of notation, we call the extended function $\mathbf{v}$. Applying the diagonalization process again, we conclude that there exists a sequence $\rho_\mu \to 0$ and a function $\mathbf{v}$ on Ω such that

$$\mathbf{v}_{\rho_\mu} \rightharpoonup \mathbf{v} \quad \text{in } W^{1,q}(\Omega_r(y))^N \text{ and in } L^{\tilde{q}}(\Omega_r(y))^N, \tag{3.2.38}$$

and

$$\mathbf{v}_{\rho_\mu} \rightharpoonup \mathbf{v} \quad \text{in } Y^{1,2}(\Omega \setminus \Omega_{r_0}(y))^N, \tag{3.2.39}$$

for all $r_0 < r < \frac{1}{2}R_y$.

Let $\phi \in C_c^\infty(\Omega)^N$ and $r < \frac{1}{2}R_y$ such that $r < d_y$. Choose $\eta \in C_c^\infty(B_r(y))$ to be a cutoff function so that $\eta \equiv 1$ in $B_{r/2}(y)$. We write $\phi = \eta\phi + (1-\eta)\phi$. By (3.2.2) and the definition of $\mathcal{B}$,

$$\lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \eta\phi^k = \lim_{\mu\to\infty} \mathcal{B}[\mathbf{v}_{\rho_\mu;y,k}, \eta\phi]$$
$$= \lim_{\mu\to\infty} \int_\Omega A_{ij}^{\alpha\beta} D_\beta \mathbf{v}^j_{\rho_\mu;y,k} D_\alpha\left(\eta\phi^i\right) + b_{ij}^\alpha \mathbf{v}^j_{\rho_\mu;y,k} D_\alpha\left(\eta\phi^i\right) + d_{ij}^\beta D_\beta \mathbf{v}^j_{\rho_\mu;y,k}\eta\phi^i + V_{ij}\mathbf{v}^j_{\rho_\mu;y,k}\eta\phi^i.$$

Note that $\eta\phi^i$ and $D\left(\eta\phi^i\right)$ belong to $C_c^\infty(\Omega_r(y))$. From this, the boundedness of $\mathbf{A}$ given by (3.1.10), and the assumptions on $\mathbf{V}$, $\mathbf{b}$, and $\mathbf{d}$ given by (3.1.11), it follows that there exists a $q' > n$ such that each of $A_{ij}^{\alpha\beta} D_\alpha\left(\eta\phi^i\right)$, $b_{ij}^\alpha D_\alpha\left(\eta\phi^i\right)$, $d_{ij}^\beta\eta\phi^i$, and $V_{ij}\eta\phi^i$ belong to $L^{q'}(\Omega_r(y))^N$. Therefore, by (3.2.38),

$$\begin{aligned}\lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \eta\phi^k &= \int_\Omega A_{ij}^{\alpha\beta} D_\beta \mathbf{v}^j_{y,k} D_\alpha\left(\eta\phi^i\right) + b_{ij}^\alpha \mathbf{v}^j_{y,k} D_\alpha\left(\eta\phi^i\right) + d_{ij}^\beta D_\beta \mathbf{v}^j_{y,k}\eta\phi^i + V_{ij}\mathbf{v}^j_{y,k}\eta\phi^i \\ &= \mathcal{B}[\mathbf{v}_{y,k}, \eta\phi]. \end{aligned} \tag{3.2.40}$$

Another application of (3.2.2) shows that

$$\begin{aligned}\lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} (1-\eta)\phi^k &= \lim_{\mu\to\infty} \int_\Omega A_{ij}^{\alpha\beta} D_\beta \mathbf{v}^j_{\rho_\mu;y,k} D_\alpha\left[(1-\eta)\phi^i\right] + b_{ij}^\alpha \mathbf{v}^j_{\rho_\mu;y,k} D_\alpha\left[(1-\eta)\phi^i\right] \\ &+ \lim_{\mu\to\infty} \int_\Omega d_{ij}^\beta D_\beta \mathbf{v}^j_{\rho_\mu;y,k}(1-\eta)\phi^i + V_{ij}\mathbf{v}^j_{\rho_\mu;y,k}(1-\eta)\phi^i.\end{aligned}$$

Since $\phi \in C_c^\infty(\Omega)^N$ and $\eta \in C_c^\infty(B_r(y))$, then $(1-\eta)\phi$ and $D[(1-\eta)\phi]$ belong to $C_c^\infty(\Omega\backslash B_{r/2}(y))^N$. In combination with (3.1.10), this implies that each $A_{ij}^{\alpha\beta} D_\alpha\left[(1-\eta)\phi^i\right]$ belongs to $L^2\left(\Omega \setminus B_{r/2}(y)\right)^N$. The assumption on $\mathbf{d}$ given in (3.1.11) implies that each $d_{ij}^\beta(1-\eta)\phi^i$ belongs to $L^2\left(\Omega \setminus B_{r/2}(y)\right)^N$ as well. And the assumption on $\mathbf{b}$ and $\mathbf{V}$ given in (3.1.11) imply that every $b_{ij}^\alpha D_\alpha\left[(1-\eta)\phi^i\right]$ and $V_{ij}(1-\eta)\phi^i$ belong to $L^{\frac{2n}{n+2}}\left(\Omega \setminus B_{r/2}(y)\right)^N$. Therefore, it follows from (3.2.39) that

$$\begin{aligned}\lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} (1-\eta)\phi^k &= \int_\Omega A_{ij}^{\alpha\beta} D_\beta \mathbf{v}^j_{y,k} D_\alpha\left[(1-\eta)\phi^i\right] + b_{ij}^\alpha \mathbf{v}^j_{y,k} D_\alpha\left[(1-\eta)\phi^i\right] \\ &+ \int_\Omega d_{ij}^\beta D_\beta \mathbf{v}^j_{y,k}(1-\eta)\phi^i + V_{ij}\mathbf{v}^j_{y,k}(1-\eta)\phi^i = \mathcal{B}[\mathbf{v}_{y,k}, (1-\eta)\phi]. \end{aligned} \tag{3.2.41}$$

It follows from combining (3.2.40) and (3.2.41) that for any $\phi \in C_c^\infty(\Omega)^N$,

$$\begin{aligned}\phi^k(y) &= \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \phi^k = \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \eta\phi^k + \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} (1-\eta)\,\phi^k \\ &= \mathcal{B}[\mathbf{v}_{y,k}, \eta\phi] + \mathcal{B}[\mathbf{v}_{y,k}, (1-\eta)\,\phi] = \mathcal{B}[\mathbf{v}_{y,k}, \phi], \end{aligned} \tag{3.2.42}$$

so that (3.2.6) holds.

As before, for any $\mathbf{f} \in L_c^\infty(\Omega)^N$, let $\mathbf{u} \in \mathbf{F}_0(\Omega)$ be the unique weak solution to $\mathcal{L}^*\mathbf{u} = \mathbf{f}$, i.e, assume that $\mathbf{u} \in \mathbf{F}_0(\Omega)$ satisfies (3.2.18). Then for a.e. $y \in \Omega$,

$$u^k(y) = \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} u^k = \lim_{\mu\to\infty} \mathcal{B}\left[\mathbf{v}_{\rho_\mu;y,k}, \mathbf{u}\right] = \lim_{\mu\to\infty} \mathcal{B}^*\left[\mathbf{u}, \mathbf{v}_{\rho_\mu;y,k}\right] = \lim_{\mu\to\infty} \int_\Omega \mathbf{v}_{\rho_\mu} \cdot \mathbf{f},$$

where we have used (3.2.2). For $\eta \in C_c^\infty(B_r(y))$ as defined in the previous paragraph, since $\eta\mathbf{f} \in L^{q'}(B_r(y))^N$ and $(1-\eta)\,\mathbf{f} \in L^{\frac{2n}{n+2}}\left(\Omega \setminus B_{r/2}(y)\right)^N$, then it follows from (3.2.38) and (3.2.39) that

$$\begin{aligned}\lim_{\mu\to\infty} \int_\Omega \mathbf{v}_{\rho_\mu} \cdot \mathbf{f} &= \lim_{\mu\to\infty} \int_{B_r(y)} \mathbf{v}_{\rho_\mu} \cdot \eta\mathbf{f} + \lim_{\mu\to\infty} \int_{\Omega\setminus B_{r/2}(y)} \mathbf{v}_{\rho_\mu} \cdot (1-\eta)\,\mathbf{f} \\ &= \int_{B_r(y)} \mathbf{v} \cdot \eta\mathbf{f} + \int_{\Omega\setminus B_{r/2}(y)} \mathbf{v} \cdot (1-\eta)\,\mathbf{f} = \int_\Omega \mathbf{v} \cdot \mathbf{f}.\end{aligned}$$

Combining the last two equations gives (3.2.7).

The estimates (3.2.8)–(3.2.13) follow almost directly by passage to the limit. Indeed, for any $r < \frac{1}{2}R_y$ and any $\mathbf{g} \in L_c^\infty(\Omega_r(y))^N$, (3.2.34) implies that

$$\left|\int_\Omega \mathbf{v} \cdot \mathbf{g}\right| = \lim_{\mu\to\infty} \left|\int_\Omega \mathbf{v}_{\rho_\mu} \cdot \mathbf{g}\right| \le C_q r^{2-n+\frac{n}{q}} \left|\left|\mathbf{g}\right|\right|_{L^{q'}(\Omega_r(y))},$$

where q' is the Hölder conjugate exponent of $q \in [1, \frac{n}{n-2})$. By duality, we obtain that for every $q \in [1, \frac{n}{n-2})$,

$$\left|\left|\mathbf{v}\right|\right|_{L^q(\Omega_r(y))} \le C_q r^{2-n+\frac{n}{q}}, \quad \forall r < \tfrac{1}{2}R_y, \tag{3.2.43}$$

that is, (3.2.10) holds. A similar argument using (3.2.33), (3.2.29) and (3.2.31), yields (3.2.11), (3.2.8), and (3.2.9), respectively. Now, as in the proofs of (3.2.30) and (3.2.32), (3.2.8) and (3.2.9) give (3.2.12) and (3.2.13).

Passing to the proof of (3.2.14), fix $x \neq y$. For a.e. $x \in \Omega$, the Lebesgue differentiation theorem implies that

$$\mathbf{v}(x) = \lim_{\delta\to 0^+} \fint_{\Omega_\delta(x)} \mathbf{v} = \lim_{\delta\to 0^+} \frac{1}{|\Omega_\delta|} \int \mathbf{v}\,\chi_{\Omega_\delta(x)},$$

where χ denotes an indicator function. Assuming as we may that $2\delta \le \min\{d_x, |x-y|\}$, it follows that $\chi_{\Omega_\delta(x)} = \chi_{B_\delta(x)} \in L^{\frac{2n}{n+2}}\left(\Omega \setminus \Omega_\delta(y)\right)$. Therefore, (3.2.39) implies that

$$\frac{1}{|B_\delta|}\int \mathbf{v}\, \chi_{B_\delta(x)} = \lim_{\mu\to\infty} \frac{1}{|B_\delta|}\int \mathbf{v}_{\rho_\mu}\, \chi_{B_\delta(x)} = \lim_{\mu\to\infty} \fint_{B_\delta(x)} \mathbf{v}_{\rho_\mu}.$$

If $|x-y| \le \frac{1}{4}R_y$ and $\rho_\mu \le \frac{1}{3}|x-y|$, $\rho_\mu < d_y$, then (3.2.23) implies that for a.e. $z \in B_\delta(x)$

$$\left|\mathbf{v}_{\rho_\mu}(z)\right| \le C\,|z-y|^{2-n},$$

where C is independent of ρ_μ. Since $|z-y| > \frac{1}{2}|x-y|$ for every $z \in B_\delta(x) \subset B_{|x-y|/2}(x)$, then

$$\left|\left|\mathbf{v}_{\rho_\mu}\right|\right|_{L^\infty(B_\delta(x))} \le C\,|x-y|^{2-n}. \tag{3.2.44}$$

On the other hand, if $|x-y| > \frac{1}{4}R_y$, then for $r := \frac{1}{8}\min\{R_x, R_y\}$, the restriction property, A1), implies that $\mathbf{v}_{\rho_\mu} \in \mathbf{F}\left(\Omega_{2r}(x)\right)$ and it follows from A4) that $\mathbf{v}_{\rho_\mu}$ vanishes along $\Omega_{2r}(x) \cap \partial\Omega$. As long as $\rho_\mu \le r$, $\rho_\mu < d_y$, $\mathcal{L}\mathbf{v}_{\rho_\mu} = \mathbf{0}$ in $\Omega_r(x)$, so we may apply (3.2.1) with $q = 2^*$. We have

$$\begin{aligned}\left|\left|\mathbf{v}_{\rho_\mu}\right|\right|_{L^\infty\left(\Omega_{r/2}(x)\right)} &\le C r^{-\frac{n-2}{2}}\left(\int_{\Omega_r(x)} \left|\mathbf{v}_{\rho_\mu}\right|^{2^*}\right)^{\frac{1}{2^*}} \\ &\le C r^{-\frac{n-2}{2}}\left(\int_{\Omega\setminus\Omega_r(y)} \left|\mathbf{v}_{\rho_\mu}\right|^{2^*}\right)^{\frac{1}{2^*}} \le C r^{2-n},\end{aligned} \tag{3.2.45}$$

where the last inequality follows from (3.2.29). If we define $R_{x,y} = \min\{R_x, R_y, |x-y|\}$, then (3.2.44) and (3.2.45) imply that for δ and ρ_μ sufficiently small (independently of each other),

$$\left|\left|\mathbf{v}_{\rho_\mu}\right|\right|_{L^\infty(B_\delta(x))} \le C R_{x,y}^{2-n}. \tag{3.2.46}$$

By combining with the observations above, we see that for a.e. $x \in \Omega$,

$$\mathbf{v}(x) = \lim_{\delta\to 0^+} \frac{1}{|\Omega_\delta|}\int \mathbf{v}\,\chi_{\Omega_\delta(x)} = \lim_{\delta\to 0^+}\lim_{\mu\to\infty} \fint_{B_\delta(x)} \mathbf{v}_{\rho_\mu} \le \lim_{\delta\to 0^+}\lim_{\mu\to\infty} C R_{x,y}^{2-n} = C R_{x,y}^{2-n}.$$

□

3.2.2 Fundamental matrix

In this section, we construct the fundamental matrix associated to $\mathcal{L}$ on $\Omega = \mathbb{R}^n$ with $n \geq 3$. We maintain the assumptions A1)–A7) with $\Omega = \mathbb{R}^n$ and replace (3.2.1) with the following global (interior) scale-invariant Moser-type bound. For the sake of future reference, within these definitions we maintain a general set Ω and emphasize their interior nature.

(IB) Let Ω be a connected open set in $\mathbb{R}^n$. We say that (IB) holds in Ω if whenever $\mathbf{u} \in \mathbf{F}(B_{2R})$ is a weak solution to $\mathcal{L}\mathbf{u} = \mathbf{f}$ or $\mathcal{L}^*\mathbf{u} = \mathbf{f}$ in B_R, for some $B_R \subset \Omega$, $R > 0$, where $\mathbf{f} \in L^\ell (B_R)^N$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$, then for any $q > 0$,

$$||\mathbf{u}||_{L^\infty\left(B_{R/2}\right)} \leq C \left[R^{-\frac{n}{q}} ||\mathbf{u}||_{L^q(B_R)} + R^{2-\frac{n}{\ell}} ||\mathbf{f}||_{L^\ell(B_R)}\right], \tag{3.2.47}$$

where the constant $C > 0$ is independent of $R > 0$.

We also assume a local Hölder continuity condition for solutions:

(H) Let Ω be a connected open set in $\mathbb{R}^n$. We say that (H) holds in Ω if whenever $\mathbf{u} \in \mathbf{F}(B_{2R_0})$ is a weak solution to $\mathcal{L}\mathbf{u} = \mathbf{0}$ or $\mathcal{L}^*\mathbf{u} = \mathbf{0}$ in B_{R_0} for some $B_{2R_0} \subset \Omega$, $R_0 > 0$, then there exists $\eta \in (0, 1)$, depending on R_0, and $C_{R_0} > 0$ so that whenever $0 < R \leq R_0$,

$$\sup_{x,y \in B_{R/2}, x \neq y} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|}{|x - y|^\eta} \leq C_{R_0} R^{-\eta} \left(\fint_{B_R} |\mathbf{u}|^{2^*} \right)^{\frac{1}{2^*}} \tag{3.2.48}$$

Notice that (IB) is (3.2.1) with $R_y = d_y$. Note also that the solutions to $\mathcal{L}\mathbf{u} = \mathbf{f}$ and $\mathcal{L}\mathbf{u} = \mathbf{0}$ above are well-defined in the weak sense for the same reason as those in (3.2.1).

The assumption (H) for equations with lower order terms implies a version of (IB) with C depending on R_0. However, since it is essential to our constructions that C in (IB) be independent of R, we require both of the assumptions (IB) and (H). In fact, determining the appropriate dependence of constants in (H) and (IB) that are sufficient for the construction of the fundamental solution was one of the biggest challenges in going from the constructions in [27] and [30] to our constructions for systems with lower order terms.

Existence of the fundamental solution may be obtained even when properties (IB) and (H) are replaced by the weaker assumption (3.2.1) (see Proposition 3.2.5). What is gained by property (IB) over (3.2.1) is a quantification of the constraint given by R_y. The property (H) assures Hölder continuity and, in addition, helps to show that $\mathbf{\Gamma}(x,y) = \mathbf{\Gamma}^*(y,x)^T$, which leads to analogous estimates for $\mathbf{\Gamma}(x,\cdot)$ as for $\mathbf{\Gamma}(\cdot,y)$.

Definition 3.2.3. *We say that the matrix function* $\mathbf{\Gamma}(x,y) = (\Gamma_{ij}(x,y))_{i,j=1}^N$ *defined on* $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ *is the* **fundamental matrix** *of* $\mathcal{L}$ *if it satisfies the following properties:*

1) $\mathbf{\Gamma}(\cdot,y)$ *is locally integrable and* $\mathcal{L}\mathbf{\Gamma}(\cdot,y) = \delta_y I$ *for all* $y \in \mathbb{R}^n$ *in the sense that for every* $\phi = \left(\phi^1,\dots,\phi^N\right)^T \in C_c^\infty(\mathbb{R}^n)^N$,

$$\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta} D_\beta \Gamma_{jk}(\cdot,y) D_\alpha \phi^i + b_{ij}^\alpha \Gamma_{jk}(\cdot,y) D_\alpha \phi^i + d_{ij}^\beta D_\beta \Gamma_{jk}(\cdot,y)\phi^i + V_{ij}\Gamma_{jk}(\cdot,y)\phi^i = \phi^k(y).$$

2) *For all* $y \in \mathbb{R}^n$ *and* $r > 0$, $\mathbf{\Gamma}(\cdot,y) \in Y^{1,2}(\mathbb{R}^n \setminus B_r(y))^{N\times N}$.

3) *For any* $\mathbf{f} = \left(f^1,\dots,f^N\right)^T \in L_c^\infty(\mathbb{R}^n)^N$, *the function* $\mathbf{u} = \left(u^1,\dots,u^N\right)^T$ *given by*

$$u^k(y) = \int_{\mathbb{R}^n} \Gamma_{jk}(x,y) f^j(x)\, dx$$

belongs to $\mathbf{F}_0(\mathbb{R}^n)$ *and satisfies* $\mathcal{L}^*\mathbf{u} = \mathbf{f}$ *in the sense that for every* $\phi = \left(\phi^1,\dots,\phi^N\right)^T \in C_c^\infty(\mathbb{R}^n)^N$,

$$\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta \phi^j + b_{ij}^\alpha D_\alpha u^i \phi^j + d_{ij}^\beta u^i D_\beta \phi^j + V_{ij} u^i \phi^j = \int_{\mathbb{R}^n} f^j \phi^j.$$

We say that the matrix function $\mathbf{\Gamma}(x,y)$ *is the* **continuous fundamental matrix** *if it satisfies the conditions above and is also continuous.*

Remark 3.2.4. As we will see below, we first establish the existence of a fundamental matrix using an application of Lemma 3.2.2. With the additional assumption of Hölder continuity of solutions, we then show that our fundamental matrix is in fact a continuous fundamental matrix.

We show here that there is at most one fundamental matrix. In general, we mean uniqueness in the sense of Lebesgue, i.e. almost everywhere uniqueness. However, when we refer to the continuous fundamental matrix, we mean true pointwise equivalence.

Assume that $\mathbf{\Gamma}$ and $\widetilde{\mathbf{\Gamma}}$ are fundamental matrices satisfying Definition 3.2.3. Then, for all $\mathbf{f} \in L_c^{\infty}(\Omega)^N$, the functions $\mathbf{u}$ and $\widetilde{\mathbf{u}}$ given by

$$u^k(y) = \int_{\mathbb{R}^n} \Gamma_{jk}(x,y)\, f^j(x)\, dx, \quad \widetilde{u}^k(y) = \int_{\mathbb{R}^n} \widetilde{\Gamma}_{jk}(x,y)\, f^j(x)\, dx$$

satisfy

$$\mathcal{L}^*(\mathbf{u} - \widetilde{\mathbf{u}}) = \mathbf{0} \quad \text{in } \mathbb{R}^n$$

and $\mathbf{u} - \widetilde{\mathbf{u}} \in \mathbf{F}_0(\mathbb{R}^n)$. By uniqueness of solutions ensured by the Lax-Milgram lemma, $\mathbf{u} - \widetilde{\mathbf{u}} \equiv \mathbf{0}$. Thus, for a.e. $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \left[\Gamma_{jk}(x,y) - \widetilde{\Gamma}_{jk}(x,y)\right] f^j(x) dx = 0, \quad \forall f \in L_c^{\infty}(\mathbb{R}^n)^N.$$

Therefore, $\mathbf{\Gamma} = \widetilde{\mathbf{\Gamma}}$ a.e. in $\{x \neq y\}$. If we further assume that $\mathbf{\Gamma}$ and $\widetilde{\mathbf{\Gamma}}$ are continuous fundamental matrices, then we conclude that $\mathbf{\Gamma} \equiv \widetilde{\mathbf{\Gamma}}$ in $\{x \neq y\}$.

Proposition 3.2.5. *Assume that* A1)–A7) *and* (3.2.1) *hold. Then there exists a fundamental matrix,* $\mathbf{\Gamma}(x,y) = (\Gamma_{ij}(x,y))_{i,j=1}^N$, $\{x \neq y\}$, *unique in the Lebesgue sense, that satisfies Definition 3.2.3. Furthermore,* $\mathbf{\Gamma}(x,y)$ *satisfies the following estimates:*

$$||\mathbf{\Gamma}(\cdot,y)||_{Y^{1,2}(\mathbb{R}^n \setminus B_r(y))} \leq C r^{1-\frac{n}{2}}, \quad \forall r < \tfrac{1}{2} R_y, \tag{3.2.49}$$

$$||\mathbf{\Gamma}(\cdot,y)||_{L^q(B_r(y))} \leq C_q r^{2-n+\frac{n}{q}}, \quad \forall q \in \left[1, \tfrac{n}{n-2}\right), \quad \forall r < \tfrac{1}{2} R_y, \tag{3.2.50}$$

$$||D\mathbf{\Gamma}(\cdot,y)||_{L^q(B_r(y))} \leq C_q r^{1-n+\frac{n}{q}}, \quad \forall q \in \left[1, \tfrac{n}{n-1}\right), \quad \forall r < \tfrac{1}{2} R_y, \tag{3.2.51}$$

$$|\{x \in \mathbb{R}^n : |\mathbf{\Gamma}(x,y)| > \tau\}| \leq C \tau^{-\frac{n}{n-2}}, \quad \forall \tau > (\tfrac{1}{2} R_y)^{2-n}, \tag{3.2.52}$$

$$|\{x \in \mathbb{R}^n : |D_x \mathbf{\Gamma}(x,y)| > \tau\}| \leq C \tau^{-\frac{n}{n-1}}, \quad \forall \tau > (\tfrac{1}{2} R_y)^{1-n}, \tag{3.2.53}$$

$$|\mathbf{\Gamma}(x,y)| \leq C R_{x,y}^{2-n}, \qquad \text{where } R_{x,y} := \min(R_x, R_y, |x-y|), \tag{3.2.54}$$

where each constant depends on n*,* N*,* c_0*,* Γ*,* γ*, and the constants from* (3.1.20) *and* (3.2.1)*, and each* C_q *depends additionally on* q*.*

Proof. By assumption, the hypotheses of Lemma 3.2.2 are satisfied, and for each $y \in \mathbb{R}^n$, $0 < \rho < d_y$, and $k = 1, \ldots, N$, we obtain $\{\mathbf{v}_{\rho;y,k}\} \subset \mathbf{F}_0(\mathbb{R}^n)$ and $\mathbf{v}_{y,k}$ satisfying properties (3.2.2)-(3.2.7) and the estimates (3.2.8)-(3.2.14).

For each $y \in \mathbb{R}^n$, define $\mathbf{\Gamma}^{\rho}(\cdot,y)$ and $\mathbf{\Gamma}(\cdot,y)$ to be the $N \times N$ matrix functions whose kth columns are given by $\mathbf{v}_{\rho;y,k}^T$ and $\mathbf{v}_{y,k}^T$, respectively. That $\mathbf{\Gamma}$ is the fundamental matrix of $\mathcal{L}$ follows immediately from the conclusions of Lemma 3.2.2. One can also deduce from Lemma 3.2.2 that $\mathbf{\Gamma}(\cdot,y)$ satisfies (3.2.49)–(3.2.54) as a function of x. $\square$

Theorem 3.2.6. *Assume that* A1)–A7) *as well as properties* (IB) *and* (H) *hold. Then there exists a unique continuous fundamental matrix,* $\mathbf{\Gamma}(x,y) = (\Gamma_{ij}(x,y))_{i,j=1}^N$, $\{x \neq y\}$, *that satisfies Definition 3.2.3. We have* $\mathbf{\Gamma}(x,y) = \mathbf{\Gamma}^*(y,x)^T$*, where* $\mathbf{\Gamma}^*$ *is the unique continuous fundamental matrix associated to* $\mathcal{L}^*$*. Furthermore,* $\mathbf{\Gamma}(x,y)$ *satisfies the following estimates:*

$$||\mathbf{\Gamma}(\cdot,y)||_{Y^{1,2}(\mathbb{R}^n \setminus B_r(y))} + ||\mathbf{\Gamma}(x,\cdot)||_{Y^{1,2}(\mathbb{R}^n \setminus B_r(x))} \le C r^{1-\frac{n}{2}}, \quad \forall r > 0, \tag{3.2.55}$$

$$||\mathbf{\Gamma}(\cdot,y)||_{L^q(B_r(y))} + ||\mathbf{\Gamma}(x,\cdot)||_{L^q(B_r(x))} \le C_q r^{2-n+\frac{n}{q}}, \quad \forall q \in \left[1, \tfrac{n}{n-2}\right), \quad \forall r > 0, \tag{3.2.56}$$

$$||D\mathbf{\Gamma}(\cdot,y)||_{L^q(B_r(y))} + ||D\mathbf{\Gamma}(x,\cdot)||_{L^q(B_r(x))} \le C_q r^{1-n+\frac{n}{q}}, \qquad \forall q \in \left[1, \tfrac{n}{n-1}\right), \quad \forall r > 0, \tag{3.2.57}$$

$$|\{x \in \mathbb{R}^n : |\mathbf{\Gamma}(x,y)| > \tau\}| + |\{y \in \mathbb{R}^n : |\mathbf{\Gamma}(x,y)| > \tau\}| \le C\tau^{-\frac{n}{n-2}}, \quad \forall \tau > 0, \tag{3.2.58}$$

$$|\{x \in \mathbb{R}^n : |D_x\mathbf{\Gamma}(x,y)| > \tau\}| + |\{y \in \mathbb{R}^n : |D_y\mathbf{\Gamma}(x,y)| > \tau\}| \le C\tau^{-\frac{n}{n-1}}, \quad \forall \tau > 0, \tag{3.2.59}$$

$$|\mathbf{\Gamma}(x,y)| \le C|x-y|^{2-n}, \qquad \forall x \neq y, \tag{3.2.60}$$

where each constant depends on n*,* N*,* c_0*,* Γ*,* γ*, and the constants from* (3.1.20) *and* (IB)*, and each* C_q *depends additionally on* q*. Moreover, for any* $0 < R \le R_0 < |x-y|$*,*

$$|\mathbf{\Gamma}(x,y) - \mathbf{\Gamma}(z,y)| \le C_{R_0} C \left(\frac{|x-z|}{R}\right)^\eta R^{2-n} \tag{3.2.61}$$

whenever $|x-z| < \frac{R}{2}$ *and*

$$|\mathbf{\Gamma}(x,y) - \mathbf{\Gamma}(x,z)| \le C_{R_0} C \left(\frac{|y-z|}{R}\right)^\eta R^{2-n} \tag{3.2.62}$$

whenever $|y-z| < \frac{R}{2}$*, where* C_{R_0} *and* $\eta = \eta(R_0)$ *are the same as in assumption* (H)*.*

Proof. By our assumptions, Proposition 3.2.5 holds with $R_y = \infty$ for all $y \in \mathbb{R}^n$. Let $\mathbf{\Gamma}^\rho(\cdot,y)$ and $\mathbf{\Gamma}(\cdot,y)$ be as in Proposition 3.2.5.

Fix x, $y \in \mathbb{R}^n$ and $0 < R \le R_0 < |x-y|$. Then $\boldsymbol{\mathcal{L}}\mathbf{\Gamma}(\cdot,y) = \mathbf{0}$ on $B_{R_0}(x)$. Therefore, by assumption (H) and the pointwise bound (3.2.54), whenever $|x-z| < \frac{R}{2}$ we have

$$|\mathbf{\Gamma}(x,y) - \mathbf{\Gamma}(z,y)| \le C_{R_0}\left(\frac{|x-z|}{R}\right)^\eta C\,||\mathbf{\Gamma}(\cdot,y)||_{L^\infty(B_R(x))} \le C_{R_0} C \left(\frac{|x-z|}{R}\right)^\eta R^{2-n}.$$

This is the Hölder continuity of $\mathbf{\Gamma}(\cdot, y)$ described by (3.2.61).

Using the pointwise bound on $\mathbf{v}_\rho$ in place of those for $\mathbf{v}$, a similar statement holds for $\mathbf{\Gamma}^\rho$ with $\rho \leq \frac{3}{8}|x-y|$, and it follows that for any compact set $K \Subset \mathbb{R}^n \setminus \{y\}$, the sequence $\{\mathbf{\Gamma}^{\rho_\mu}(\cdot, y)\}_{\mu=1}^{\infty}$ is equicontinuous on K. Furthermore, for any such $K \Subset \mathbb{R}^n \setminus \{y\}$, there are constants $C_K < \infty$ and $\rho_K > 0$ such that for all $\rho < \rho_K$,

$$||\mathbf{\Gamma}^\rho(\cdot, y)||_{L^\infty(K)} \leq C_K.$$

Passing to a subsequence if necessary, we have that for any such compact $K \Subset \mathbb{R}^n \setminus \{y\}$,

$$\mathbf{\Gamma}^{\rho_\mu}(\cdot, y) \to \mathbf{\Gamma}(\cdot, y) \tag{3.2.63}$$

uniformly on K.

We now aim to show

$$\mathbf{\Gamma}(x, y) = \mathbf{\Gamma}^*(y, x)^T,$$

where $\mathbf{\Gamma}^*$ is the fundamental matrix associated to $\mathcal{L}^*$. Let $\widehat{\mathbf{v}}_\sigma = \widehat{\mathbf{v}}_{\sigma;x,k}$ denote the averaged fundamental vector from Lemma 3.2.2 associated to $\mathcal{L}^*$. By the same arguments used for $\mathbf{v}_\rho$, we obtain a sequence $\{\sigma_\nu\}_{\nu=1}^{\infty}$, $\sigma_\nu \to 0$, such that $\widehat{\mathbf{\Gamma}}^{\sigma_\nu}(\cdot, x)$, a matrix whose k-th column is $\widehat{\mathbf{v}}_{\sigma_\nu;x,k}^T$, converges to $\mathbf{\Gamma}^*(\cdot, x)$ uniformly on compact subsets of $\mathbb{R}^n \setminus \{x\}$, where $\mathbf{\Gamma}^*(\cdot, x)$ is a fundamental matrix for $\mathcal{L}^*$ that satisfies the properties analogous to those for $\mathbf{\Gamma}(\cdot, y)$. In particular, $\mathbf{\Gamma}^*(\cdot, x)$ is Hölder continuous.

By (3.2.2), for ρ_μ and σ_ν sufficiently small,

$$\fint_{B_\rho(y)} \widehat{\Gamma}^\sigma_{kl}(\cdot, x) = \mathcal{B}\left[\mathbf{v}_{\rho;y,l}, \widehat{\mathbf{v}}_{\sigma;x,k}\right] = \mathcal{B}^*\left[\widehat{\mathbf{v}}_{\sigma;x,k}, \mathbf{v}_{\rho;y,l}\right] = \fint_{B_\sigma(x)} \Gamma^\rho_{lk}(\cdot, y). \tag{3.2.64}$$

Define

$$g^{kl}_{\mu\nu} := \fint_{B_{\rho_\mu}(y)} \widehat{\Gamma}^{\sigma_\nu}_{kl}(\cdot, x) = \fint_{B_{\sigma_\nu}(x)} \Gamma^{\rho_\mu}_{lk}(\cdot, y).$$

By continuity of $\Gamma^{\rho_\mu}_{lk}(\cdot, y)$, it follows that for any $x \neq y \in \mathbb{R}^n$,

$$\lim_{\nu\to\infty} g^{kl}_{\mu\nu} = \lim_{\nu\to\infty} \fint_{B_{\rho_\mu}(y)} \widehat{\Gamma}^{\sigma_\nu}_{kl}(\cdot, x) = \Gamma^{\rho_\mu}_{lk}(x, y),$$

so that by (3.2.63),

$$\lim_{\mu\to\infty}\lim_{\nu\to\infty} g^{kl}_{\mu\nu} = \lim_{\mu\to\infty} \Gamma^{\rho_\mu}_{lk}(x, y) = \Gamma_{lk}(x, y).$$

But by weak convergence in $W^{1,q}(B_r(y))$, i.e., (3.2.3) with $R_y = \infty$,

$$\lim_{\nu\to\infty} g^{kl}_{\mu\nu} = \lim_{\nu\to\infty} \fint_{B_{\rho_\mu}(y)} \widehat{\Gamma}^{\sigma_\nu}_{kl}(\cdot, x) = \fint_{B_{\rho_\mu}(y)} \Gamma^*_{kl}(\cdot, x),$$

and it follows then by continuity of $\Gamma^*_{kl}(\cdot, x)$ that

$$\lim_{\mu\to\infty}\lim_{\nu\to\infty} g^{kl}_{\mu\nu} = \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \Gamma^*_{kl}(\cdot, x) = \Gamma^*_{kl}(y, x).$$

Therefore, for all $k, l \in \{1, \ldots, N\}$, $x \neq y$,

$$\Gamma_{lk}(x, y) = \Gamma^*_{kl}(y, x),$$

or equivalently, for all $x \neq y$,

$$\mathbf{\Gamma}(x, y) = \mathbf{\Gamma}^*(y, x)^T. \tag{3.2.65}$$

Consequently, all the estimates which hold for $\mathbf{\Gamma}(\cdot, y)$ hold analogously for $\mathbf{\Gamma}(x, \cdot)$. $\square$

Remark 3.2.7. We have seen that there is a subsequence $\{\rho_\mu\}_{\mu=1}^{\infty}$, $\rho_\mu \to 0$, such that $\mathbf{\Gamma}^{\rho_\mu}(x, y) \to \mathbf{\Gamma}(x, y)$ for all $x \in \mathbb{R}^n \setminus \{y\}$. In fact, a stronger fact can be proved. By (3.2.64),

$$\Gamma^{\rho}_{lk}(x, y) = \lim_{\nu\to\infty} \fint_{B_{\sigma_\nu}(x)} \Gamma^{\rho}_{lk}(\cdot, y) = \lim_{\nu\to\infty} \fint_{B_\rho(y)} \widehat{\Gamma}^{\sigma_\nu}_{kl}(\cdot, x) = \fint_{B_\rho(y)} \Gamma^*_{kl}(\cdot, x).$$

By (3.2.65), this gives

$$\Gamma^{\rho}_{lk}(x, y) = \fint_{B_\rho(y)} \Gamma_{lk}(x, z)\, dz.$$

By continuity, for all $x \neq y$,

$$\lim_{\rho\to 0} \mathbf{\Gamma}^{\rho}(x, y) = \mathbf{\Gamma}(x, y). \tag{3.2.66}$$

Theorem 3.2.8. *Assume that* A1)–A7) *as well as properties* (IB) *and* (H) *hold. If* $\mathbf{f} \in \left(L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^{\ell}_{\mathrm{loc}}(\mathbb{R}^n)\right)^N$ *for some* $\ell \in \left(\frac{n}{2}, \infty\right]$, *then there exists a unique* $\mathbf{u} \in \mathbf{F}_0(\mathbb{R}^n)$ *that is a weak solution to* $\mathcal{L}\mathbf{u} = \mathbf{f}$. *Furthermore, we have*

$$u^k(x) = \int_{\mathbb{R}^n} \Gamma_{ki}(x, y)\, f^i(y)\, dy, \quad k = 1, \ldots, N. \tag{3.2.67}$$

for a.e. $x \in \mathbb{R}^n$.

Proof. We see from (3.2.17) that

$$\mathbf{F}_0(\mathbb{R}^n) \ni \mathbf{w} \mapsto \int_{\mathbb{R}^n} \mathbf{f} \cdot \mathbf{w}$$

defines a bounded linear functional on $\mathbf{F}_0(\mathbb{R}^n)$. Therefore, the existence of a unique $\mathbf{u} \in \mathbf{F}_0(\mathbb{R}^n)$ that is a weak solution to $\mathcal{L}\mathbf{u} = \mathbf{f}$ follows from the Lax-Milgram theorem.

By definition of a weak solution, we have

$$\int_{\mathbb{R}^n} \mathbf{f} \cdot \widehat{\mathbf{v}}_\sigma = \mathcal{B}[\mathbf{u}, \widehat{\mathbf{v}}_\sigma] = \mathcal{B}^*[\widehat{\mathbf{v}}_\sigma, \mathbf{u}] = \fint_{\Omega_\sigma(x)} u^k, \tag{3.2.68}$$

where $\widehat{\mathbf{v}}_\sigma = \widehat{\mathbf{v}}_{\sigma;\,y,k}$ is the averaged fundamental vector from Lemma 3.2.2 associated to $\mathcal{L}^*$. Taking the limit in σ of the left-hand side, we get

$$\lim_{\sigma\to 0} \int_{\mathbb{R}^n} \mathbf{f} \cdot \widehat{\mathbf{v}}_\sigma = \lim_{\sigma\to 0} \left(\int_{B_1(x)} \mathbf{f} \cdot \widehat{\mathbf{v}}_\sigma + \int_{\mathbb{R}^n \setminus B_1(x)} \mathbf{f} \cdot \widehat{\mathbf{v}}_\sigma \right) = \int_{\mathbb{R}^n} \mathbf{f} \cdot \widehat{\mathbf{v}}, \tag{3.2.69}$$

where $\widehat{\mathbf{v}}$ is the k-th column of $\mathbf{\Gamma}^*(\cdot, x)$. Here, we have used (3.2.22) and $\mathbf{f} \in L^\ell_{\text{loc}}(\mathbb{R}^n)^N$ for $\ell \in \left(\frac{n}{2}, \infty\right]$ to establish convergence of the first integral, and we have used (3.2.5) and $\mathbf{f} \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)^N$ to establish convergence of the second integral. Combining (3.2.68) and (3.2.69), we get

$$u^k(x) = \int_{\mathbb{R}^n} \Gamma^*_{ik}(y, x) f^i(y)\, dy, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

The conclusion (3.2.67) now follows from (3.2.65). □

3.2.3 Green matrix

Here we show existence of the Green matrix of $\mathcal{L}$ on any connected open set $\Omega \subset \mathbb{R}^n$ with $n \geq 3$.

Definition 3.2.9. *Let Ω be an open, connected subset of $\mathbb{R}^n$. We say that the matrix function $\mathbf{G_D}(x,y) = (G_{ij}(x,y))_{i,j=1}^N$ defined on the set $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ is the* ***Green matrix*** *of $\mathcal{L}$ if it satisfies the following properties:*

1) $\mathbf{G_D}(\cdot, y)$ is locally integrable and $\mathcal{L}\mathbf{G_D}(\cdot, y) = \delta_y I$ for all $y \in \Omega$ in the sense that for every $\phi = \left(\phi^1, \ldots, \phi^N\right)^T \in C_c^\infty(\Omega)^N$,

$$\int_\Omega A^{\alpha\beta}_{ij} D_\beta G_{jk}(\cdot, y) D_\alpha \phi^i + b^\alpha_{ij} G_{jk}(\cdot, y) D_\alpha \phi^i + d^\beta_{ij} D_\beta G_{jk}(\cdot, y)\phi^i + V_{ij} G_{jk}(\cdot, y)\phi^i$$
$$= \phi^k(y)$$

2) *For all $y \in \Omega$ and $r > 0$, $\mathbf{G_D}(\cdot, y) \in Y^{1,2}(\Omega \setminus \Omega_r(y))^{N \times N}$. In addition, $\mathbf{G_D}(\cdot, y)$ vanishes on $\partial\Omega$ in the sense that for every $\zeta \in C_c^\infty(\Omega)$ satisfying $\zeta \equiv 1$ on $B_r(y)$ for some $r > 0$, we have*

$$(1-\zeta)\mathbf{G_D}(\cdot, y) \in Y_0^{1,2}(\Omega \setminus \Omega_r(y))^{N \times N}.$$

3) *For any $\mathbf{f} = \left(f^1, \ldots, f^N\right)^T \in L_c^\infty(\Omega)^N$, the function $\mathbf{u} = \left(u^1, \ldots, u^N\right)^T$ given by*

$$u^k(y) = \int_\Omega G_{jk}(x, y) f^j(x)\, dx$$

belongs to $\mathbf{F}_0(\Omega)$ and satisfies $\mathcal{L}^*\mathbf{u} = \mathbf{f}$ in the sense that for every $\phi = \left(\phi^1, \ldots, \phi^N\right)^T \in C_c^\infty(\Omega)^N$,

$$\int_\Omega A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta \phi^j + b_{ij}^\alpha D_\alpha u^i \phi^j + d_{ij}^\beta u^i D_\beta \phi^j + V_{ij} u^i \phi^j = \int_\Omega f^j \phi^j.$$

We say that the matrix function $\mathbf{G_D}(x, y)$ is the ***continuous Green matrix*** *if it satisfies the conditions above and is also continuous.*

As in the case of the (continuous) fundamental matrix, and by the same argument, there exists at most one (continuous) Green matrix, where the sense of uniqueness is also as before.

Theorem 3.2.10. *Let Ω be an open, connected, proper subset of $\mathbb{R}^n$. Denote $d_x :=$* dist$(x, \partial\Omega)$ *for $x \in \Omega$. Assume that* A1)–A7) *as well as properties* (IB) *and* (H) *hold. Then there exists a unique continuous Green matrix $\mathbf{G_D}(x, y) = (G_{ij}(x, y))_{i,j=1}^N$, defined in $\{x, y \in \Omega, x \neq y\}$, that satisfies Definition 3.2.9. We have $\mathbf{G_D}(x, y) = \mathbf{G_D}^*(y, x)^T$, where $\mathbf{G_D}^*$ is the unique continuous Green matrix associated to $\mathcal{L}^*$. Furthermore,*

$\mathbf{G_D}(x,y)$ *satisfies the following estimates:*

$$||\mathbf{G_D}(\cdot,y)||_{Y^{1,2}(\Omega\setminus B_r(y))} \le Cr^{1-n/2}, \qquad \forall r < \tfrac{1}{2}d_y, \tag{3.2.70}$$

$$||\mathbf{G_D}(\cdot,y)||_{L^q(B_r(y))} \le C_q r^{2-n+\frac{n}{q}}, \qquad \forall r < \tfrac{1}{2}d_y, \quad \forall q \in [1, \tfrac{n}{n-2}), \tag{3.2.71}$$

$$||D\mathbf{G_D}(\cdot,y)||_{L^q(B_r(y))} \le C_q r^{1-n+\frac{n}{q}}, \qquad \forall r < \tfrac{1}{2}d_y, \quad \forall q \in [1, \tfrac{n}{n-1}), \tag{3.2.72}$$

$$|\{x\in\Omega : |\mathbf{G_D}(x,y)| > \tau\}| \le C\tau^{-\frac{n}{n-2}}, \qquad \forall \tau > \left(\tfrac{1}{2}d_y\right)^{2-n}, \tag{3.2.73}$$

$$|\{x\in\Omega : |D_x\mathbf{G_D}(x,y)| > \tau\}| \le C\tau^{-\frac{n}{n-1}}, \qquad \forall \tau > \left(\tfrac{1}{2}d_y\right)^{1-n}, \tag{3.2.74}$$

$$||\mathbf{G_D}(x,\cdot)||_{Y^{1,2}(\Omega\setminus B_r(x))} \le Cr^{1-n/2}, \qquad \forall r < \tfrac{1}{2}d_x, \tag{3.2.75}$$

$$||\mathbf{G_D}(x,\cdot)||_{L^q(B_r(x))} \le C_q r^{2-n+\frac{n}{q}}, \qquad \forall r < \tfrac{1}{2}d_x, \quad \forall q \in [1, \tfrac{n}{n-2}), \tag{3.2.76}$$

$$||D\mathbf{G_D}(x,\cdot)||_{L^q(B_r(x))} \le C_q r^{1-n+\frac{n}{q}}, \qquad \forall r < \tfrac{1}{2}d_x, \quad \forall q \in [1, \tfrac{n}{n-1}), \tag{3.2.77}$$

$$|\{y\in\Omega : |\mathbf{G_D}(x,y)| > \tau\}| \le C\tau^{-\frac{n}{n-2}}, \qquad \forall \tau > \left(\tfrac{1}{2}d_x\right)^{2-n}, \tag{3.2.78}$$

$$|\{y\in\Omega : |D_y\mathbf{G_D}(x,y)| > \tau\}| \le C\tau^{-\frac{n}{n-1}}, \qquad \forall \tau > \left(\tfrac{1}{2}d_x\right)^{1-n}, \tag{3.2.79}$$

$$|\mathbf{G_D}(x,y)| \le Cd_{x,y}^{2-n} \qquad \forall x\ne y, \text{ where } d_{x,y} := min(d_x, d_y, |x-y|), \tag{3.2.80}$$

where the constants depend on n, N, c_0, Γ, γ, *and the constants from* (3.1.20) *and* (IB), *and each* C_q *depends additionally on* q. *Moreover, for any* $0 < R \le R_0 < \frac{1}{2}d_{x,y}$,

$$|\mathbf{G_D}(x,y) - \mathbf{G_D}(z,y)| \le C_{R_0}C\left(\frac{|x-z|}{R}\right)^{\eta} R^{2-n}, \tag{3.2.81}$$

whenever $|x-z| < \frac{R}{2}$ *and*

$$|\mathbf{G_D}(x,y) - \mathbf{G_D}(x,z)| \le C_{R_0}C\left(\frac{|y-z|}{R}\right)^{\eta} R^{2-n}, \tag{3.2.82}$$

whenever $|y-z| < \frac{R}{2}$, *where* C_{R_0} *and* $\eta = \eta(R_0)$ *are the same as in assumption* (H).

Proof. The hypotheses of Lemma 3.2.2 are satisfied with $R_y = d_y$, for all $y \in \Omega$. For each $y\in\Omega$, $0<\rho<d_y$, and $k = 1,\dots,N$, we obtain $\{\mathbf{v}_{\rho;y,k}\} \subset \mathbf{F}_0(\Omega)$ and $\mathbf{v} = \mathbf{v}_{y,k}$ satisfying (3.2.2)-(3.2.7) and the estimates (3.2.8)-(3.2.14), where $R_y = d_y$ and $R_{x,y} = \min\{d_x, d_y, |x-y|\}$.

We define $\mathbf{G_D}(\cdot,y)$ to be the matrix whose columns are given by $\mathbf{v}_{y,k}^T$ for $k = 1,\dots,N$, and we define similarly the averaged Green matrix $\mathbf{G_D}^{\rho}(\cdot,y)$. Then estimates (3.2.70)–(3.2.74) and (3.2.80) are inherited directly from Lemma 3.2.2.

We now prove that $\mathbf{G_D}(x,y)$ satisfies Definition 3.2.9. This definition largely resembles that of the fundamental matrix, and the proof can be executed analogously, except for an additional requirement to prove that $\mathbf{G_D}(\cdot, y) = \mathbf{0}$ on $\partial\Omega$ in the sense that for all $\zeta \in C_c^\infty(\Omega)$ satisfying $\zeta \equiv 1$ on $B_r(y)$ for some $r > 0$, we have

$$(1-\zeta)\mathbf{G_D}(\cdot, y) \in Y_0^{1,2}(\Omega)^{N\times N}. \tag{3.2.83}$$

By Mazur's lemma, $Y_0^{1,2}(\Omega)^N$ is weakly closed in $Y^{1,2}(\Omega)^N$. Therefore, since $(1-\zeta)\mathbf{v}_{\rho_\mu} = \mathbf{v}_{\rho_\mu} - \zeta\mathbf{v}_{\rho_\mu} \in Y_0^{1,2}(\Omega)^N$ for all $\rho_\mu < d_y$, it suffices for (3.2.83) to show that

$$(1-\zeta)\mathbf{v}_{\rho_\mu} \rightharpoonup (1-\zeta)\mathbf{v} \quad \text{in } Y^{1,2}(\Omega)^N. \tag{3.2.84}$$

Since $(1-\zeta) \equiv 0$ on $B_r(y)$, the result (3.2.84) follows from (3.2.5). Indeed,

$$\begin{aligned}
\int_\Omega (1-\zeta)G_{kl}(\cdot, y)\phi &= \int_\Omega G_{kl}(\cdot, y)(1-\zeta)\phi = \lim_{\mu\to\infty}\int_\Omega G_{kl}^{\rho_\mu}(\cdot, y)(1-\zeta)\phi \\
&= \lim_{\mu\to\infty}\int_\Omega (1-\zeta)G_{kl}^{\rho_\mu}(\cdot, y)\phi, \quad \forall \phi \in L^{\frac{2n}{n+2}}(\Omega), \quad \text{and} \\
\int_\Omega D\left[(1-\zeta)\, G_{kl}(\cdot, y)\right]\cdot\psi &= -\int_\Omega G_{kl}(\cdot, y)D\zeta\cdot\psi + \int_\Omega DG_{kl}(\cdot, y)\cdot(1-\zeta)\psi \\
&= -\lim_{\mu\to\infty}\int_\Omega G_{kl}^{\rho_\mu}(\cdot, y)D\zeta\cdot\psi + \lim_{\mu\to\infty}\int_\Omega DG_{kl}^{\rho_\mu}(\cdot, y)\cdot(1-\zeta)\psi \\
&= \lim_{\mu\to\infty}\int_\Omega D\left[(1-\zeta)\, G_{kl}^{\rho_\mu}(\cdot, y)\right]\cdot\psi, \quad \forall \psi \in L^2(\Omega)^N.
\end{aligned}$$

Therefore, $\mathbf{G_D}(x, y)$ is the unique Green matrix associated to $\mathcal{L}$.

It follows from (3.2.70) and property (H) that for any $0 < R \le R_0 \le \frac{1}{2}d_{x,y}$, there exists $\eta = \eta(R_0)$ and $C_{R_0} > 0$ such that, whenever $|x - z| \le \frac{R}{2}$,

$$|\mathbf{G_D}(x, y) - \mathbf{G_D}(z, y)| \le C_{R_0} C \left(\frac{|x-z|}{R}\right)^\eta R^{2-n}. \tag{3.2.85}$$

By the same argument that lead to (3.2.63), this implies that, passing to a subsequence if necessary, for any compact $K \Subset \Omega \setminus \{y\}$,

$$\mathbf{G_D}^{\rho_\mu}(\cdot, y) \to \mathbf{G_D}(\cdot, y) \tag{3.2.86}$$

uniformly on K, and from here the same argument as the one for (3.2.65) proves that

$$\mathbf{G_D}(x, y) = \mathbf{G_D}^*(y, x)^T, \quad \forall x, y \in \Omega, \quad x \neq y. \tag{3.2.87}$$

The remaining properties, (3.2.75)–(3.2.79), follow from Lemma 3.2.2 applied to $\mathbf{G_D}^*(\cdot, x)$ in combination with (3.2.87). $\square$

Remark 3.2.11. As with the fundamental matrix, we obtain

$$\mathbf{G_D}^{\rho}(x,y) = \fint_{\Omega_\rho(y)} \mathbf{G_D}(x,z)dz, \tag{3.2.88}$$

and, by continuity,

$$\lim_{\rho \to 0} \mathbf{G_D}^{\rho}(x,y) = \mathbf{G_D}(x,y), \quad \forall x, y \in \Omega, \quad x \neq y. \tag{3.2.89}$$

3.2.4 Global estimates for the Green matrix

It was observed in [30] that if the interior boundedness assumption (IB) is altered as below (to being valid on balls possibly intersecting the boundary), then the pointwise and local L^q estimates of $\mathbf{G_D}$ can be freed of their dependence on the distances to the boundary for the homogeneous elliptic operators. Similarly, assuming local boundedness on boundary balls gives enhanced Green function estimates in our setting.

(BB) Let Ω be a connected open set in $\mathbb{R}^n$. We say that (BB) holds in Ω if whenever $\mathbf{u} \in \mathbf{F}(\Omega_{2R})$ is a weak solution to $\mathcal{L}\mathbf{u} = \mathbf{f}$ or $\mathcal{L}^*\mathbf{u} = \mathbf{f}$ in Ω_R, for some $R > 0$, where $\mathbf{f} \in L^\ell(\Omega_R)^N$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$, and $\mathbf{u} \equiv \mathbf{0}$ on $\partial\Omega \cap B_R$, then $\mathbf{u}$ is a bounded function and for any $q > 0$,

$$||\mathbf{u}||_{L^\infty(\Omega_{R/2})} \leq C\left[R^{-\frac{n}{q}} ||\mathbf{u}||_{L^q(\Omega_R)} + R^{2-\frac{n}{\ell}} ||\mathbf{f}||_{L^\ell(\Omega_R)}\right], \tag{3.2.90}$$

where the constant C is independent of R.

We note that condition (BB) holds, for example, whenever (IB) holds for an extended operator $\mathcal{L}^\#$ defined on $\mathbb{R}^n$ with $\mathcal{L} = \mathcal{L}^\#$ on Ω. This fact can often be established by a reflection argument (see, for example, Appendix A).

Corollary 3.2.12. *Let Ω be an open, connected, proper subset of $\mathbb{R}^n$. Assume that* A1)–A7) *as well as properties* (BB) *and* (H) *hold. Then the continuous Green matrix* *satisfies the following global estimates:*

$$||\mathbf{G_D}(\cdot,y)||_{Y^{1,2}(\Omega\setminus B_r(y))}+||\mathbf{G_D}(x,\cdot)||_{Y^{1,2}(\Omega\setminus B_r(x))}\le Cr^{1-n/2},\qquad\forall r>0,\tag{3.2.91}$$

$$||\mathbf{G_D}(\cdot,y)||_{L^q(B_r(y))}+||\mathbf{G_D}(x,\cdot)||_{L^q(B_r(x))}\le C_q r^{2-n+\frac{n}{q}},\qquad\forall r>0,\quad\forall q\in[1,\tfrac{n}{n-2}),\tag{3.2.92}$$

$$||D\mathbf{G_D}(\cdot,y)||_{L^q(B_r(y))}+||D\mathbf{G_D}(x,\cdot)||_{L^q(B_r(x))}\le C_q r^{1-n+\frac{n}{q}},\qquad\forall r>0,\quad\forall q\in[1,\tfrac{n}{n-1}),\tag{3.2.93}$$

$$|\{x\in\Omega:|\mathbf{G_D}(x,y)|>\tau\}|+|\{y\in\Omega:|\mathbf{G_D}(x,y)|>\tau\}|\le C\tau^{-\frac{n}{n-2}},\qquad\forall\tau>0,\tag{3.2.94}$$

$$|\{x\in\Omega:|D_x\mathbf{G_D}(x,y)|>\tau\}|+|\{y\in\Omega:|D_y\mathbf{G_D}(x,y)|>\tau\}|\le C\tau^{-\frac{n}{n-1}},\qquad\forall\tau>0,\tag{3.2.95}$$

$$|\mathbf{G_D}(x,y)|\le C|x-y|^{2-n}\qquad\forall x\ne y,\tag{3.2.96}$$

where the constants depend on n, N, c_0, Γ, γ *and the constants from* (3.1.20) *and* (BB)*, and each* C_q *depends additionally on* q*. The Hölder continuity estimates of Theorem 3.2.10 remain unchanged.*

Proof. As in the proof of Theorem 3.2.10, the global estimates are inherited directly from Lemma 3.2.2 with $R_x, R_y = \infty$ for all $x, y \in \Omega$. □

Remark 3.2.13. In conclusion, $\mathbf{\Gamma}(x,y)$ exists and satisfies the estimates of Theorem 3.2.6 whenever (IB) and (H) hold for solutions. The conclusion of Theorem 3.2.6 also states that

$$\mathbf{\Gamma}(\cdot,y)\in Y^{1,2}\left(\mathbb{R}^n\setminus B_r(y)\right)^{N\times N}\quad\text{for any } r>0.$$

However, it does not follow from Theorem 3.2.6 that $\mathbf{\Gamma}(\cdot,y)\in\mathbf{F}(\mathbb{R}^n\setminus B_r(y))$ for the general space $\mathbf{F}$. In Section 3.6, we examine a number of examples and show that in each case, a version of this statement holds for $\mathbf{\Gamma}(\cdot,y)$ as well as $\mathbf{\Gamma}(x,\cdot)$, $\mathbf{G_D}(\cdot,y)$, and $\mathbf{G_D}(x,\cdot)$.

3.3 A Caccioppoli inequality

The remainder of the chapter will essentially be a discussion of the major examples that fit our theory. In this section we prove a version of the Caccioppoli inequality designed to satisfy hypothesis A7). In the next two sections we demonstrate local boundedness and Hölder continuity of solutions (for equations only, rather than systems) in the spirits of properties (BB) and (H). And finally, in Section 3.6, we tie it all together by presenting the most common examples.

Lemma 3.3.1. *Let $\Omega \subset \mathbb{R}^n$ be open and connected. Assume that $\mathbf{F}(\Omega)$, $\mathbf{F}_0(\Omega)$, $\mathcal{L}$, and $\mathcal{B}$ satisfy* A1) – A5). Suppose $\mathbf{b} \in L^s(\Omega)^{n\times N\times N}$, $\mathbf{d} \in L^t(\Omega)^{n\times N\times N}$ for some $s, t \in [n, \infty]$, and (instead of assuming A6)), assume that either

- $s, t = n$ and $\mathcal{B}[\mathbf{v}, \mathbf{v}] \geq \gamma \, ||D\mathbf{v}||^2_{L^2(\Omega)^N}$ for every $\mathbf{v} \in \mathbf{F}_0(\Omega)$; or
- $s, t \in (n, \infty]$ and $\mathcal{B}[\mathbf{v}, \mathbf{v}] \geq \gamma \, ||\mathbf{v}||^2_{W^{1,2}(\Omega)^N}$ for every $\mathbf{v} \in \mathbf{F}_0(\Omega)$.

Let $\mathbf{u} \in \mathbf{F}(\Omega)$ and $\zeta \in C^\infty(\mathbb{R}^n)$ with $D\zeta \in C_c^\infty(\mathbb{R}^n)$ be such that $\mathbf{u}\zeta \in \mathbf{F}_0(\Omega)$, $D_\alpha\zeta\, \mathbf{u} \in L^2(\Omega)^N$, $\alpha = 1, ..., n$, and $\mathcal{B}\left[\mathbf{u}, \mathbf{u}\zeta^2\right] \leq \int \mathbf{f} \cdot \mathbf{u}\,\zeta^2$ for some $\mathbf{f} \in L^\ell(\Omega)^N$, $\ell \in \left(\frac{n}{2}, \infty\right]$. Then

$$\int |D\mathbf{u}|^2 \zeta^2 \leq C \int |\mathbf{u}|^2 |D\zeta|^2 + c \left| \int \mathbf{f} \cdot \mathbf{u}\,\zeta^2 \right|, \tag{3.3.1}$$

where $C = C\left(n, s, t, \gamma, \Lambda, ||\mathbf{b}||_{L^s(\Omega)}, ||\mathbf{d}||_{L^t(\Omega)}\right)$, $c = c(\gamma)$.

Remark 3.3.2. Let us make a few comments before the proof. First, as in the comments to A7), we remark that the condition $D\zeta \in C_c^\infty(\mathbb{R}^n)$ implies that ζ is a constant outside some large ball (call it C_ζ) and hence, $C_\zeta - \zeta \in C_c^\infty(\mathbb{R}^n)$. Then, by A4), $\mathbf{u}\zeta^2 = C_\zeta \mathbf{u}\zeta - (C_\zeta - \zeta)\mathbf{u}\zeta \in \mathbf{F}_0(\Omega)$. We shall use this in the proof. Also, the conditions $\mathbf{u}\zeta \in \mathbf{F}_0(\Omega)$, $D_\alpha\zeta\,\mathbf{u} \in L^2(\Omega)^N$, $\alpha = 1, ..., n$, and $D\zeta \in C_c^\infty(\mathbb{R}^n)$, along with (3.1.2), ensure that the first and the second integral in (3.3.1) are finite. The last one is finite for otherwise both the assumptions and the conclusion of the Lemma are meaningless.

Second, if we do assume A6), then the condition $\mathcal{B}[\mathbf{v}, \mathbf{v}] \geq \gamma \, ||D\mathbf{v}||^2_{L^2(\Omega)^N}$ for every $\mathbf{v} \in \mathbf{F}_0(\Omega)$ follows from (3.1.2). Moreover, the actual requirements on $\mathbf{b}$ and $\mathbf{d}$ that are necessary to carry out the arguments, and appear in the constant C, are $\mathbf{b} \in L^s(\Omega \cap U)^{n\times N\times N}$, $\mathbf{d} \in L^t(\Omega \cap U)^{n\times N\times N}$ for any U containing the support of $D\zeta$. Since the latter is compact, one could always reduce the case $s, t > n$ to the case $s = t = n$ and hence to work in the first regimen. However, such a reduction would bring up the dependence of the constants on the size of the support of $D\zeta$, and this is typically not desirable.

Proof. Let $\mathbf{u}$, ζ be as in the statement. A computation shows that

$$\mathcal{B}\left[\mathbf{u}\zeta, \mathbf{u}\zeta\right] = \mathcal{B}\left[\mathbf{u}, \mathbf{u}\zeta^2\right] + \int \mathbf{A}^{\alpha\beta}\left[\left(-D_\beta \mathbf{u}\cdot \mathbf{u}\, D_\alpha\zeta + \mathbf{u}\, D_\beta\zeta \cdot D_\alpha \mathbf{u}\right)\zeta + \mathbf{u}\cdot\mathbf{u}\, D_\beta\zeta\, D_\alpha\zeta\right]$$
$$+ \int \left(-\mathbf{b}^\alpha \mathbf{u}\zeta \cdot \mathbf{u}\, D_\alpha\zeta + \mathbf{d}^\beta \mathbf{u} D_\beta\zeta\cdot\mathbf{u}\zeta\right).$$

By the assumption, $\mathcal{B}\left[\mathbf{u}, \mathbf{u}\zeta^2\right] \leq \int |\mathbf{f}|\,|\mathbf{u}|\,\zeta^2$.
By (3.1.10),

$$\int \mathbf{A}^{\alpha\beta}\left[\left(-D_\beta \mathbf{u}\cdot \mathbf{u}\, D_\alpha\zeta + \mathbf{u}\, D_\beta\zeta \cdot D_\alpha \mathbf{u}\right)\zeta + \mathbf{u}\cdot\mathbf{u}\, D_\beta\zeta\, D_\alpha\zeta\right]$$
$$\leq 2\Lambda \int |D\mathbf{u}|\,|D\zeta|\,|\mathbf{u}|\,\eta + \Lambda\int |D\zeta|^2\,|\mathbf{u}|^2 \leq \left(\frac{8\Lambda^2}{\gamma} + \Lambda\right)\int |\mathbf{u}|^2\,|D\zeta|^2 + \frac{\gamma}{8}\int |D\mathbf{u}|^2\,\zeta^2.$$

If $s \in (n, \infty)$, then since $\mathbf{u}\zeta \in \mathbf{F}_0(\Omega)$,

$$\left|\int \mathbf{b}^\alpha \mathbf{u}\zeta\cdot\mathbf{u}\, D_\alpha\zeta\right| \leq \int |\mathbf{b}|\,|\mathbf{u}\zeta|^{\frac{n}{s}}\,|\mathbf{u}\zeta|^{1-\frac{n}{s}}\,|\mathbf{u}D\zeta| \leq ||\mathbf{b}||_{L^s(\Omega)}\,||\mathbf{u}\zeta||^{\frac{n}{s}}_{L^{2^*}(\Omega)}\,||\mathbf{u}\zeta||^{1-\frac{n}{s}}_{L^2(\Omega)}\,||\mathbf{u}D\zeta||_{L^2(\Omega)}$$
$$\leq c_n^{\frac{n}{s}}\,||\mathbf{b}||_{L^s(\Omega)}\,||D(\mathbf{u}\zeta)||^{\frac{n}{s}}_{L^2(\Omega)}\,||\mathbf{u}\zeta||^{1-\frac{n}{s}}_{L^2(\Omega)}\,||\mathbf{u}D\zeta||_{L^2(\Omega)}$$
$$\leq \frac{\gamma}{4}\,||D(\mathbf{u}\zeta)||^2_{L^2(\Omega)} + \frac{\gamma}{2}\,||\mathbf{u}\zeta||^2_{L^2(\Omega)} + \frac{C_{n,s}}{\gamma}\,||\mathbf{b}||^2_{L^s(\Omega)}\int |\mathbf{u}|^2\,|D\zeta|^2.$$

Similarly, if $s = \infty$, then

$$\left|\int \mathbf{b}^\alpha \mathbf{u}\zeta\cdot\mathbf{u}\, D_\alpha\zeta\right| \leq ||\mathbf{b}||_{L^\infty(\Omega)}\,||\mathbf{u}\zeta||_{L^2(\Omega)}\,||\mathbf{u}D\zeta||_{L^2(\Omega)} \leq \frac{\gamma}{2}\,||\mathbf{u}\zeta||^2_{L^2(\Omega)} + \frac{1}{2\gamma}\,||\mathbf{b}||^2_{L^\infty(\Omega)}\int |\mathbf{u}|^2\,|D\zeta|^2.$$

Finally, if $s = n$, then

$$\left|\int \mathbf{b}^\alpha \mathbf{u}\zeta\cdot\mathbf{u}\, D_\alpha\zeta\right| \leq \int |\mathbf{b}|\,|\mathbf{u}\zeta|\,|\mathbf{u}D\zeta| \leq ||\mathbf{b}||_{L^n(\Omega)}\,||\mathbf{u}\zeta||_{L^{2^*}(\Omega)}\,||\mathbf{u}D\zeta||_{L^2(\Omega)}$$
$$\leq \frac{\gamma}{4}\int |D(\mathbf{u}\zeta)|^2 + \frac{c_n^2}{\gamma}\,||\mathbf{b}||^2_{L^n(\Omega)}\int |\mathbf{u}|^2\,|D\zeta|^2.$$

Analogous inequalities hold for $\mathbf{d}$.

It follows from the inequalities above and the coercivity assumption on $\mathcal{B}$ that

$$\frac{\gamma}{4}\int |D\mathbf{u}|^2\,\zeta^2 - \frac{\gamma}{2}\int |\mathbf{u}|^2\,|D\zeta|^2 \leq \frac{\gamma}{2}\int |D(\mathbf{u}\zeta)|^2$$
$$\leq \left(\frac{8\Lambda^2}{\gamma} + \Lambda + \frac{C_{n,s}\,||\mathbf{b}||^2_{L^s(\Omega)}}{\gamma} + \frac{C_{n,t}\,||\mathbf{d}||^2_{L^t(\Omega)}}{\gamma}\right)\int |\mathbf{u}|^2\,|D\zeta|^2 + \frac{\gamma}{8}\int |D\mathbf{u}|^2\,\zeta^2 + \int |\mathbf{f}|\,|\mathbf{u}|\,\zeta^2,$$

which leads to the claimed inequality after rearrangements. $\square$

3.4 Local boundedness in the equation setting

For general elliptic systems, homogeneous or not, (IB), (BB), (H), or even the fact of local boundedness of solutions may fail. For counterexamples, we refer to [38] for dimension $n \geq 5$ and [20] for lower dimensions. In this and the next section we discuss the cases when local boundedness is valid, restricting ourselves to the context of equations rather than systems, i.e., to $N = 1$. We insist that such a restriction is taken in Sections 3.4 and 3.5 only and that this restriction is not necessary in order for (IB), (BB), (H) to hold. Nonetheless, it is perhaps the most commonly used application. Much of the material in Sections 3.4 and 3.5, or at least analogous arguments, have appeared in classical literature (e.g., [22], [24], [47]). However, we have to carefully track the constants, the exact nature of dependence on $\mathbf{b}$, $\mathbf{d}$, $\mathbf{V}$, the impact of coercivity, and the resulting scale-invariance, since this is crucial for building the fundamental solutions. Therefore, for completeness, we present the full arguments.

The following lemma gives a scale-invariant (independent of the choice of R) version of local boundedness. To prove the lemma, we will use de Giorgi's approach, as explained in [24], [47]. The novelty of our argument is that rather than assuming ellipticity of the homogeneous operator, we assume coercivity of the bilinear form associated to the full operator. This allows us to prove a scale-invariant version of local boundedness under a certain sign assumption on the lower order terms. In other words, we avoid picking up dependencies on the size of the domain over which we are working. Recall that $\Omega_R = B_R \cap \Omega$.

We continue to work in the abstract framework that was first introduced in Section 3.1, but we will have to impose some further properties on our function spaces in order to show that local boundedness and interior Hölder continuity are in fact reasonable assumptions.

B1) For any $R > 0$, $k \geq 0$, if $u \in \mathbf{F}(\Omega_R)$ satisfies $u = 0$ along $\partial\Omega \cap B_R$ (as usual, in the sense of (3.1.7)–(3.1.8)), then $\zeta (u-k)_+ \in \mathbf{F}_0(\Omega_R)$, $D_\alpha \zeta (u-k)_+ \in L^2(\Omega_R)$, $\alpha = 1, ..., n$, for any non-negative $\zeta \in C_c^\infty(B_R)$, where $(u-k)_+ := \max\{u-k, 0\}$.

B2) For any ball $B_R \subset \mathbb{R}^n$, $R > 0$, if $u \in \mathbf{F}(B_R)$ is non-negative, and $k, \omega > 0$, then $(u+k)^{-\omega} \in \mathbf{F}(B_R)$.

Lemma 3.4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and connected and take $N = 1$. Assume that $\mathbf{F}(\Omega)$, $\mathbf{F}_0(\Omega)$, $\mathcal{L}$, and $\mathcal{B}$ satisfy* A1) – A5) and B1). Suppose $b \in L^s(\Omega_R)^n$, $d \in L^t(\Omega_R)^n$ for some $s, t \in [n, \infty]$, and (instead of assuming A6)) assume that either

- $s, t = n$ and $\mathcal{B}[v, v] \geq \gamma \, ||Dv||^2_{L^2(\Omega_R)}$ for every $v \in \mathbf{F}_0(\Omega_R)$; or
- $s, t \in (n, \infty]$ and $\mathcal{B}[v, v] \geq \gamma \, ||v||^2_{W^{1,2}(\Omega_R)}$ for every $v \in \mathbf{F}_0(\Omega_R)$.

Assume also that

$$V - D_\alpha b^\alpha \geq 0 \text{ in } \Omega_R \text{ in the sense of distributions.} \tag{3.4.1}$$

Let $u \in \mathbf{F}(\Omega_{2R})$ satisfy $u = 0$ along $\partial\Omega \cap B_{2R}$. Let $f \in L^\ell(\Omega_R)$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$ and assume that $\mathcal{L}u \leq f$ in Ω_R weakly in the sense that for any $\varphi \in \mathbf{F}_0(B_R)$ such that $\varphi \geq 0$ in Ω_R, we have

$$\mathcal{B}[u, \varphi] \leq \int f\varphi. \tag{3.4.2}$$

Then $u^+ \in L^\infty_{loc}(\Omega_R)$ and for any $r < R$, $q > 0$,

$$\sup_{\Omega_r} u^+ \leq \frac{C}{(R-r)^{\frac{n}{q}}} \left|\left|u^+\right|\right|_{L^q(\Omega_R)} + c_q R^{2-\frac{n}{\ell}} \, ||f||_{L^\ell(\Omega_R)}, \tag{3.4.3}$$

where $C = C\left(n, q, s, t, \ell, \gamma, \Lambda, ||b||_{L^s(\Omega_R)}, ||d||_{L^t(\Omega_R)}\right)$ and c_q depends only on q.

Remark 3.4.2. Let us remark that (3.4.3) is of course vacuous if $||u^+||_{L^q(\Omega_R)}$ is not finite. In practice, however, this is not a concern because in any ball of radius strictly smaller than R, the norm is finite and hence we can apply (3.4.3) in such a ball. Indeed, $||u^+||_{L^q(\Omega_R)} < \infty$ for any $u \in \mathbf{F}(\Omega_{2R})$ by (3.1.5). Therefore, by applying (3.4.3) with $q = 2$, we conclude that $||u^+||_{L^\infty(\Omega_r)}$ is finite for any $r < R$. Hence, $||u^+||_{L^q(\Omega_r)}$ is finite for any $r < R$. Below, we will first prove (3.4.3) with $q = 2$ and then assume that $||u^+||_{L^q(\Omega_R)}$ is finite. (Again, one can always take a slightly smaller ball if necessary).

Remark 3.4.3. If $\Omega_R = B_R$, then $\partial\Omega \cap \Omega_R$ is empty so that the boundary condition on u is vacuously satisfied. Therefore, this version of local boundedness is applicable for all of our settings, i.e. when we are concerned with the boundary and when we are not.

Remark 3.4.4. As previously pointed out, the estimate (3.4.3) is scale-invariant since it doesn't depend on R. In our applications, we will assume that $b \in L^s(\Omega)^n$ and

$d \in L^t(\Omega)^n$. Since $||b||_{L^s(\Omega_R)} \le ||b||_{L^s(\Omega)}$ and $||d||_{L^t(\Omega_R)} \le ||d||_{L^t(\Omega)}$ for every R, then this lemma shows that we may establish local bounds with constants that are independent of the subdomain, Ω_R.

Proof. We will first prove the case of $q = 2$ and $r = \frac{1}{2}R$. Fix $\zeta \in C_c^\infty(B_R)$, a cutoff function for which $0 \le \zeta \le 1$. For some $k \ge 0$, define $v = (u-k)_+$. By B1), $v\zeta, v\zeta^2 \in \mathbf{F}_0(\Omega_R)$. Lemma 7.6 from [22] implies that $Dv = Du$ for $u > k$ and $Dv = 0$ for $u \le k$ (since (3.1.5) implies that v is weakly differentiable on Ω).

Since $V - D_\alpha b^\alpha \ge 0$ in the sense of distributions and $\text{supp}(v\zeta^2)$ is a subset of $\{u \ge k\}$, then

$$\begin{aligned}\mathcal{B}\left[v, v\zeta^2\right] &= \int \left(A^{\alpha\beta} D_\beta v + b^\alpha v\right) D_\alpha\left(v\zeta^2\right) + \left(d^\beta D_\beta v + Vv\right) v\zeta^2 \\ &= \mathcal{B}\left[u, v\zeta^2\right] - k\int \left(V - D_\alpha b^\alpha\right) v\zeta^2 \le \int f v\zeta^2,\end{aligned}$$

where we used (3.4.2) with $\varphi := v\zeta^2 \in \mathbf{F}_0(\Omega_R)$, $\varphi \ge 0$ to get the last inequality.

Since $v\zeta \in \mathbf{F}_0(\Omega_R)$, $vD_\alpha\zeta \in L^2(\Omega_R)$, and $D\zeta$ is compactly supported, then Lemma 3.3.1 is applicable with $\mathbf{u} = v$. It follows that

$$\begin{aligned}\int |Dv|^2\zeta^2 \le \left[\left(\frac{8\Lambda}{\gamma}\right)^2 + \frac{8\Lambda}{\gamma} + 4 + 8\frac{C_{n,s}\,||b||^2_{L^s(\Omega_R)} + C_{n,t}\,||d||^2_{L^t(\Omega_R)}}{\gamma^2}\right]\int |v|^2|D\zeta|^2 \\ + \frac{8}{\gamma}\int |f|\,|v|\,\zeta^2.\end{aligned}$$

By Hölder and Sobolev inequalities with $2^* = \frac{2n}{n-2}$,

$$\begin{aligned}\int |f|\,v\zeta^2 &\le \left(\int |f|^\ell\right)^{\frac{1}{\ell}}\left(\int |v\zeta|^{2^*}\right)^{\frac{1}{2^*}} |\{v\zeta \ne 0\}|^{1-\frac{1}{\ell}-\frac{1}{2^*}} \\ &\le \frac{\gamma}{32}\int |D(v\zeta)|^2 + \frac{8c_n^2}{\gamma}\,||f||^2_{L^\ell(\Omega_R)}\,|\{v\zeta \ne 0\}|^{1+\frac{2}{n}-\frac{2}{\ell}}.\end{aligned} \tag{3.4.4}$$

Therefore,

$$\begin{aligned}\int |D(v\zeta)|^2 \le 4\left[\left(\frac{8\Lambda}{\gamma}\right)^2 + \frac{8\Lambda}{\gamma} + 5 + 8\frac{C_{n,s}\,||b||^2_{L^s(\Omega_R)} + C_{n,t}\,||d||^2_{L^t(\Omega_R)}}{\gamma^2}\right]\int |v|^2|D\zeta|^2 \\ + \left(\frac{16c_n}{\gamma}\,||f||_{L^\ell(\Omega_R)}\right)^2 |\{v\zeta \ne 0\}|^{1+\frac{2}{n}-\frac{2}{\ell}}.\end{aligned}$$

Since the Hölder and Sobolev inequalities imply that

$$\int (v\zeta)^2 \leq \left(\int (v\zeta)^{2^*}\right)^{2/2^*} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}} \leq c_n^2 \int |D(v\zeta)|^2 |\{v\zeta \neq 0\}|^{\frac{2}{n}},$$

then

$$\int (v\zeta)^2 \leq \frac{C_1}{4} \int v^2 |D\zeta|^2 |\{v\zeta \neq 0\}|^{\varepsilon_1} + C_1 F^2 |\{v\zeta \neq 0\}|^{1+\varepsilon_2}, \tag{3.4.5}$$

where $\varepsilon_1 = \frac{2}{n}$, $\varepsilon_2 = \frac{4}{n} - \frac{2}{\ell} > 0$, $F = ||f||_{L^\ell(\Omega_R)}$, and

$$C_1 = 16c_n^2 \left[\left(\frac{8\Lambda}{\gamma}\right)^2 + \frac{8\Lambda}{\gamma} + 5 + 8\frac{C_{n,s}\,||b||^2_{L^s(\Omega_R)} + C_{n,t}\,||d||^2_{L^t(\Omega_R)}}{\gamma^2}\right] + \left(\frac{16c_n^2}{\gamma}\right)^2. \tag{3.4.6}$$

For fixed $0 < r \leq \rho \leq R$, let $\zeta \in C_c^\infty(B_\rho)$ be such that $\zeta \equiv 1$ in B_r and $|D\zeta| \leq \frac{2}{\rho - r}$ in B_R. We let $A(k,r) = \{x \in \Omega_r : u \geq k\} = \operatorname{supp} v \cap \Omega_r$. Then, for any $0 < r < \rho \leq R$ and $k \geq 0$, it follows from (3.4.5) that

$$\int_{A(k,r)} (u-k)^2 \leq C_1 \left[\frac{|A(k,\rho)|^{\varepsilon_1}}{(\rho - r)^2} \int_{A(k,\rho)} (u-k)^2 + F^2 |A(k,\rho)|^{1+\varepsilon_2}\right]. \tag{3.4.7}$$

Considering $r = R/2$, the goal is to show that there exists a $k \geq 0$ such that

$$\int_{A(k,R/2)} (u-k)^2 = 0.$$

Take $h > k \geq 0$ and $0 < r < R$. Since $A(k,r) \supset A(h,r)$, then

$$\int_{A(h,r)} (u-h)^2 \leq \int_{A(k,r)} (u-k)^2$$

and

$$|A(h,r)| = |B_r \cap \{u - k > h - k\}| \leq \frac{1}{(h-k)^2} \int_{A(k,r)} (u-k)^2.$$

Using these inequalities in (3.4.7) above, we have that for $h > k \geq 0$ and $\frac{1}{2}R \leq r < \rho \leq R$

$$\begin{aligned} \int_{A(h,r)} (u-h)^2 &\leq C_1 \left[\frac{|A(h,\rho)|^{\varepsilon_1}}{(\rho - r)^2} \int_{A(h,\rho)} (u-h)^2 + F^2 |A(h,\rho)|^{1+\varepsilon_2}\right] \\ &\leq C_1 \Bigg\{ \frac{1}{(\rho - r)^2 (h-k)^{2\varepsilon_1}} \left[\int_{A(k,\rho)} (u-k)^2\right]^{1+\varepsilon_1} \\ &\quad + \frac{F^2}{(h-k)^{2(1+\varepsilon_2)}} \left[\int_{A(k,\rho)} (u-k)^2\right]^{1+\varepsilon_2} \Bigg\} \end{aligned}$$

or

$$\left|\left|(u-h)^+\right|\right|_{L^2(\Omega_r)} \leq C_2 \left[\frac{1}{(\rho - r)(h-k)^{\varepsilon_1}} \left|\left|(u-k)^+\right|\right|^{1+\varepsilon_1}_{L^2(\Omega_\rho)} + \frac{F}{(h-k)^{1+\varepsilon_2}} \left|\left|(u-k)^+\right|\right|^{1+\varepsilon_2}_{L^2(\Omega_\rho)}\right], \tag{3.4.8}$$

where C_2 depends on C_1.

Set $\varphi(k,r) = \left|\left|(u-k)^+\right|\right|_{L^2(\Omega_r)}$. For $i = 0, 1, 2, \ldots$, define

$$k_i = K\left(1 - \frac{1}{2^i}\right), \qquad r_i = \frac{R}{2} + \frac{R}{2^{i+1}},$$

so that

$$k_i - k_{i-1} = \frac{K}{2^i}, \qquad r_{i-1} - r_i = \frac{R}{2^{i+1}},$$

where $K > 0$ is to be determined. Then it follows from (3.4.8) with $\rho = r_{i-1}$, $r = r_i$, $h = k_i$, and $k = k_{i-1}$ that

$$\varphi(k_i, r_i) \leq C_2 \left[2\frac{2^{(1+\varepsilon_1)i}}{RK^{\varepsilon_1}} \varphi(k_{i-1}, r_{i-1})^{1+\varepsilon_1} + F\left(\frac{2^i}{K}\right)^{1+\varepsilon_2} \varphi(k_{i-1}, r_{i-1})^{1+\varepsilon_2}\right], \quad i \geq 1. \tag{3.4.9}$$

Claim: There exists $\mu > 1$ and K sufficiently large (depending, in particular, on μ) such that for any $i = 0, 1, \ldots$

$$\varphi(k_i, r_i) \leq \frac{\varphi(k_0, r_0)}{\mu^i}. \tag{3.4.10}$$

It is clear that the claim holds for $i = 0$. Assume that the claim holds for $i - 1$. Then

$$\varphi(k_{i-1}, r_{i-1})^{1+\varepsilon} \leq \left[\frac{\varphi(k_0, r_0)}{\mu^{i-1}}\right]^{1+\varepsilon} = \left[\frac{\varphi(k_0, r_0)^\varepsilon}{\mu^{i\varepsilon - (1+\varepsilon)}}\right] \frac{\varphi(k_0, r_0)}{\mu^i}.$$

Substituting this expression into (3.4.9), we have

$$\begin{aligned}\varphi(k_i, r_i) \leq C_2 \Bigg[& 2\mu^{(1+\varepsilon_1)} \left(\frac{2^{(1+\varepsilon_1)}}{\mu^{\varepsilon_1}}\right)^i \left[\frac{\varphi(k_0, r_0)}{R^{\frac{n}{2}} K}\right]^{\varepsilon_1} \\ & + \mu^{(1+\varepsilon_2)} \left(\frac{2^{(1+\varepsilon_2)}}{\mu^{\varepsilon_2}}\right)^i \frac{R^{\frac{n}{2}\varepsilon_2} F}{K} \left[\frac{\varphi(k_0, r_0)}{R^{\frac{n}{2}} K}\right]^{\varepsilon_2}\Bigg] \frac{\varphi(k_0, r_0)}{\mu^i}.\end{aligned}$$

If we choose $\mu > 1$ so that $\mu^{\varepsilon_i} \geq 2^{2+\varepsilon_i}$ for each i, then for the claim to hold we need

$$C_2 \left\{ 2\mu^{(1+\varepsilon_1)} \left[\frac{R^{-\frac{n}{2}} \varphi\left(k_0, r_0\right)}{K} \right]^{\varepsilon_1} + \mu^{(1+\varepsilon_2)} \frac{R^{2-\frac{n}{\ell}} F}{K} \left[\frac{R^{-\frac{n}{2}} \varphi\left(k_0, r_0\right)}{K} \right]^{\varepsilon_2} \right\} \leq 1.$$

Thus, we choose $K = C_0 R^{-n/2} \varphi\left(k_0, r_0\right) + R^{2-\frac{n}{\ell}} F$ for some $C_0 >> 1$ that depends on C_2, μ and each ε_i.

Taking $i \to \infty$ in (3.4.10) shows that $\varphi\left(K, \frac{R}{2}\right) = 0$. In other words, since $\varphi\left(k_0, r_0\right) = \varphi\left(0, R\right) = ||u^+||_{L^2(\Omega_R)}$,

$$\sup_{\Omega_{R/2}} u^+ \leq K \leq C_0 R^{-\frac{n}{2}} \left|\left|u^+\right|\right|_{L^2(\Omega_R)} + R^{2-\frac{n}{\ell}} \left|\left|f\right|\right|_{L^\ell(\Omega_R)}.$$

For any $q \in [2, \infty]$, an application of the Hölder inequality gives

$$\sup_{\Omega_{R/2}} u^+ \leq C_0 R^{-\frac{n}{q}} \left|\left|u^+\right|\right|_{L^q(\Omega_R)} + R^{2-\frac{n}{\ell}} \left|\left|f\right|\right|_{L^\ell(\Omega_R)}. \tag{3.4.11}$$

To obtain an estimate in $\Omega_{\theta R}$, we apply (3.4.11) to $\Omega_{(1-\theta)R}\left(y\right)$, where $y \in \Omega_{\theta R}$. That is, for any $y \in \Omega_{\theta R}$,

$$u^+\left(y\right) \leq \sup_{\Omega_{\frac{(1-\theta)R}{2}}(y)} u^+ \leq C_0 \left[\left(1-\theta\right) R\right]^{-\frac{n}{q}} \left|\left|u^+\right|\right|_{L^q(\Omega_R)} + R^{2-\frac{n}{\ell}} \left|\left|f\right|\right|_{L^\ell(\Omega_R)}.$$

Now for $\theta \in (0,1)$, $R > 0$, and $q \in (0, 2)$, we have

$$\begin{aligned}
\left|\left|u^+\right|\right|_{L^\infty(\Omega_{\theta R})} &\leq C_0 \left[\left(1-\theta\right) R\right]^{-\frac{n}{2}} \left|\left|u^+\right|\right|_{L^2(\Omega_R)} + R^{2-\frac{n}{\ell}} \left|\left|f\right|\right|_{L^\ell(\Omega_R)} \\
&\leq C_0 \left[\left(1-\theta\right) R\right]^{-\frac{n}{2}} \left|\left|u^+\right|\right|^{1-\frac{q}{2}}_{L^\infty(\Omega_R)} \left[\int_{\Omega_R} \left(u^+\right)^q\right]^{\frac{1}{2}} + R^{2-\frac{n}{\ell}} \left|\left|f\right|\right|_{L^\ell(\Omega_R)} \\
&\leq \frac{1}{2} \left|\left|u^+\right|\right|_{L^\infty(\Omega_R)} + C_{0,q} \left[\left(1-\theta\right) R\right]^{-\frac{n}{q}} \left[\int_{\Omega_R} \left(u^+\right)^q\right]^{\frac{1}{q}} + R^{2-\frac{n}{\ell}} \left|\left|f\right|\right|_{L^\ell(\Omega_R)},
\end{aligned}$$

where $C_{0,q}$ depends on q and C_0. Assuming that $||u^+||_{L^\infty(\Omega_R)} < \infty$ (recall the remark before the proof), set $h\left(t\right) = ||u^+||_{L^\infty(\Omega_t)}$ for $t \in (0, R]$. Then, for $\theta \in (0,1)$, $R > 0$, and $q \in (0, 2)$, we have

$$h\left(r\right) \leq \frac{1}{2} h\left(R\right) + \frac{C_{0,q}}{\left(R-r\right)^{\frac{n}{q}}} \left|\left|u^+\right|\right|_{L^q(\Omega_R)} + R^{2-\frac{n}{\ell}} \left|\left|f\right|\right|_{L^\ell(\Omega_R)}.$$

It follows from Lemma 4.3 in [24] that for any $r < R$,

$$\left|\left|u^+\right|\right|_{L^\infty(\Omega_r)} \leq c_q \left[\frac{C_{0,q}}{\left(R-r\right)^{\frac{n}{q}}} \left|\left|u^+\right|\right|_{L^q(\Omega_R)} + R^{2-\frac{n}{\ell}} \left|\left|f\right|\right|_{L^\ell(\Omega_R)}\right].$$

$\square$

Remark 3.4.5. If u is a supersolution, then the conclusions of the previous lemmas apply to u^- in place of u^+.

Now we prove a slightly different version of Moser boundedness. We show that without the assumptions of coercivity and non-degeneracy, solutions are still locally bounded, but there is a dependence on the size of the domain and on the negative part of the zeroth order potential.

Lemma 3.4.6. *Let* $\Omega \subset \mathbb{R}^n$ *be open and connected and take* $N = 1$. *Assume that* $\mathbf{F}(\Omega)$, $\mathbf{F}_0(\Omega)$, $\mathcal{L}$, *and* $\mathcal{B}$ *satisfy* A1) – A5) and B1). Suppose $V = V_+ - V_-$ where $V_\pm \geq 0$ a.e. and $V_- \in L^p(\Omega_R)$ for some $p \in \left(\frac{n}{2}, \infty\right]$. Assume that $b \in L^s(\Omega_R)^n$, $d \in L^t(\Omega_R)^n$ for some $s, t \in (n, \infty]$. Let $u \in \mathbf{F}(\Omega_{2R})$ satisfy $u = 0$ along $\partial\Omega \cap B_{2R}$. Let $f \in L^\ell(\Omega_R)$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$ and assume that $\mathcal{L}u \leq f$ in Ω_R weakly in the sense that for any $\varphi \in \mathbf{F}_0(B_R)$ such that $\varphi \geq 0$ in Ω_R, we have (3.4.2). Then $u^+ \in L^\infty_{loc}(\Omega_R)$ and for any $r < R$, $q > 0$, (3.4.3) holds with C dependent on n, q, p, s, t, ℓ, λ, Λ, $R^{2-\frac{n}{p}} ||V_-||_{L^p(\Omega_R)}$, $R^{1-\frac{n}{s}} ||b||_{L^s(\Omega_R)}$, $R^{1-\frac{n}{t}} ||d||_{L^t(\Omega_R)}$, where c_q depends only on q.

Proof. We will first prove the case of $q = 2$, $R = 1$, and $r = \frac{1}{2}$. Fix $\zeta \in C_c^\infty(B_1)$, a cutoff function for which $0 \leq \zeta \leq 1$. For some $k \geq 0$, define $v = (u-k)_+$. By B1), $v\zeta, v\zeta^2 \in \mathbf{F}_0(\Omega_1)$, and $Dv = Du$ for $u > k$, $Dv = 0$ for $u \leq k$, by Lemma 7.6 from [22] (since (3.1.5) implies that v is weakly differentiable on Ω).

Since $\mathrm{supp}(v\zeta^2)$ is a subset of $\{u \geq k\}$, then a computation gives

$$\int A^{\alpha\beta} D_\beta v D_\alpha v \zeta^2 = \mathcal{B}\left[u, v\zeta^2\right] - 2\int A^{\alpha\beta} D_\beta v D_\alpha \zeta v \zeta - \int \left[b^\alpha v D_\alpha\left(v\zeta^2\right) + \left(d^\beta D_\beta v + Vv\right) v\zeta^2\right] - k\int \left[b^\alpha D_\alpha\left(v\zeta^2\right) + Vv\zeta^2\right].$$

Therefore,

$$\begin{aligned}\int A^{\alpha\beta} D_\beta(v\zeta) D_\alpha(v\zeta) &= \mathcal{B}\left[u, v\zeta^2\right] + \int A^{\alpha\beta}\left(D_\beta\zeta D_\alpha v - D_\beta v D_\alpha\zeta\right) v\zeta + \int A^{\alpha\beta} D_\beta\zeta D_\alpha\zeta |v|^2 \\ &\quad - \int \left[b^\alpha v D_\alpha\left(v\zeta^2\right) + \left(d^\beta D_\beta v + Vv\right) v\zeta^2\right] - k\int \left[b^\alpha D_\alpha\left(v\zeta^2\right) + Vv\zeta^2\right] \\ &\leq \int (f + kV_-) v\zeta^2 - k\int b^\alpha D_\alpha(v\zeta)\zeta - k\int b^\alpha D_\alpha\zeta v\zeta \\ &\quad + \int A^{\alpha\beta}\left(D_\beta\zeta D_\alpha v - D_\beta v D_\alpha\zeta\right) v\zeta + \int A^{\alpha\beta} D_\beta\zeta D_\alpha\zeta |v|^2 \\ &\quad - \int \left(b^\alpha - d^\alpha\right) D_\alpha\zeta v^2\zeta - \int \left(b^\alpha + d^\alpha\right) D_\alpha(v\zeta) v\zeta + \int V_- v^2\zeta^2,\end{aligned}$$

where we used (3.4.2) with $\varphi := v\zeta^2 \in \mathbf{F}_0(\Omega_1)$, $\varphi \geq 0$ to get the first term in the last inequality. An application of the Hölder, Sobolev, and Young inequalities shows that

$$\int b^\alpha D_\alpha \zeta v^2 \zeta \leq \frac{\lambda}{4}\int |D(v\zeta)|^2 + \frac{16c_n^2}{\lambda}\,||b||^2_{L^s(\Omega_1)}\int v^2\,|D\zeta|^2\,|\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{2}{s}}.$$

Similarly,

$$\int b^\alpha D_\alpha(v\zeta)\,v\zeta \leq c_n\,||b||_{L^s(\Omega_1)}\int |D(v\zeta)|^2\,|\{v\zeta \neq 0\}|^{\frac{1}{n}-\frac{1}{s}}$$

$$\int V_- v^2\zeta^2 \leq c_n^2\,||V_-||_{L^p(\Omega_1)}\int |D(v\zeta)|^2\,|\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{p}}.$$

The ellipticity condition, (3.1.9), in combination with boundedness (3.1.10) and the computations above, shows that

$$\begin{aligned}\int |D(v\zeta)|^2 \leq{}& \frac{4}{\lambda}\int (f+kV_-)\,v\zeta^2 - \frac{4k}{\lambda}\int b^\alpha D_\alpha(v\zeta)\,\zeta - \frac{4k}{\lambda}\int b^\alpha D_\alpha\zeta v\zeta\\ &+\frac{4}{\lambda^2}\left[8\Lambda^2+\lambda\Lambda+\frac{\lambda^2}{4}+16c_n^2\left(||b||^2_{L^s(\Omega_1)}\,|\{v\zeta\neq 0\}|^{\frac{2}{n}-\frac{2}{s}}+||d||^2_{L^t(\Omega_1)}\,|\{v\zeta\neq 0\}|^{\frac{2}{n}-\frac{2}{t}}\right)\right]\int v^2\,|D\zeta|^2\\ &+\frac{4c_n}{\lambda}\Bigg(||b||_{L^s(\Omega_1)}\,|\{v\zeta\neq 0\}|^{\frac{1}{n}-\frac{1}{s}}+||d||_{L^t(\Omega_1)}\,|\{v\zeta\neq 0\}|^{\frac{1}{n}-\frac{1}{t}}\\ &\quad+c_n\,||V_-||_{L^p(\Omega_1)}\,|\{v\zeta\neq 0\}|^{\frac{2}{n}-\frac{1}{p}}\Bigg)\int |D(v\zeta)|^2.\end{aligned}$$

As in (3.4.4),

$$\int f v\zeta^2 \leq \frac{\lambda}{32}\int |D(v\zeta)|^2 + \frac{8c_n^2}{\lambda}\,||f||^2_{L^\ell(\Omega_1)}\,|\{v\zeta\neq 0\}|^{1+\frac{2}{n}-\frac{2}{\ell}}$$

$$\int V_- v\zeta^2 \leq \frac{\lambda}{32k}\int |D(v\zeta)|^2 + \frac{8kc_n^2}{\lambda}\,||V_-||^2_{L^p(\Omega_1)}\,|\{v\zeta\neq 0\}|^{1+\frac{2}{n}-\frac{2}{p}}.$$

Similarly,

$$\int b^\alpha D_\alpha(v\zeta)\,\zeta \leq \frac{\lambda}{32k}\int |D(v\zeta)|^2 + \frac{8k}{\lambda}\,||b||^2_{L^s(\Omega_1)}\,|\{v\zeta\neq 0\}|^{1-\frac{2}{s}}$$

$$\int b^\alpha D_\alpha\zeta v\zeta \leq \frac{\lambda}{32k}\int v^2\,|D\zeta|^2 + \frac{8k}{\lambda}\,||b||^2_{L^s(\Omega_1)}\,|\{v\zeta\neq 0\}|^{1-\frac{2}{s}}.$$

It follows that

$$\int |D(v\zeta)|^2 \leq \left(\frac{8c_n}{\lambda} ||f||_{L^\ell(\Omega_1)}\right)^2 |\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{2}{\ell}}$$
$$+ k^2 \left[\left(\frac{8c_n}{\lambda} ||V_-||_{L^p(\Omega_1)}\right)^2 |\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{2}{p}} + 2\left(\frac{8}{\lambda} ||b||_{L^s(\Omega_1)}\right)^2 |\{v\zeta \neq 0\}|^{1-\frac{2}{s}}\right]$$
$$+ \frac{8}{\lambda^2}\left[8\Lambda^2 + \lambda\Lambda + \frac{\lambda^2}{4} + 16c_n^2\left(||b||^2_{L^s(\Omega_1)} |\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{2}{s}} + ||d||^2_{L^t(\Omega_1)} |\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{2}{t}}\right)\right] \int v^2 |D\zeta|^2$$
$$+ \frac{8c_n}{\lambda}\Bigg(||b||_{L^s(\Omega_1)} |\{v\zeta \neq 0\}|^{\frac{1}{n}-\frac{1}{s}} + ||d||_{L^t(\Omega_1)} |\{v\zeta \neq 0\}|^{\frac{1}{n}-\frac{1}{t}}$$
$$+ c_n ||V_-||_{L^p(\Omega_1)} |\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{p}}\Bigg) \int |D(v\zeta)|^2 .$$

If $|\{v\zeta \neq 0\}|$ is chosen so that

$$|\{v\zeta \neq 0\}| \leq \min\left\{\left(\frac{\lambda}{32c_n ||b||_{L^s(\Omega_1)}}\right)^{\frac{ns}{s-n}}, \left(\frac{\lambda}{32c_n ||d||_{L^t(\Omega_1)}}\right)^{\frac{nt}{t-n}}, \left(\frac{\lambda}{32c_n^2 ||V_-||_{L^p(\Omega_1)}}\right)^{\frac{np}{2p-n}}\right\} \tag{3.4.12}$$

then

$$\int |D(v\zeta)|^2 \leq \left(\frac{256\Lambda^2}{\lambda^2} + \frac{32\Lambda}{\lambda} + \frac{9\lambda^2}{8}\right) \int v^2 |D\zeta|^2 + \left(\frac{16c_n}{\lambda} ||f||_{L^\ell(\Omega_1)}\right)^2 |\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{2}{\ell}}$$
$$+ \left(\frac{16kc_n}{\lambda} ||V_-||_{L^p(\Omega_1)}\right)^2 |\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{2}{p}} + 2\left(\frac{16k}{\lambda} ||b||_{L^s(\Omega_1)}\right)^2 |\{v\zeta \neq 0\}|^{1-\frac{2}{s}} .$$

Since the Hölder and Sobolev inequalities imply that

$$\int (v\zeta)^2 \leq \left(\int (v\zeta)^{2^*}\right)^{2/2^*} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}} \leq c_n^2 \int |D(v\zeta)|^2 |\{v\zeta \neq 0\}|^{\frac{2}{n}} ,$$

then

$$\int (v\zeta)^2 \leq \frac{C_1}{4} \int v^2 |D\zeta|^2 |\{v\zeta \neq 0\}|^{\varepsilon} + C_1 (F+k)^2 |\{v\zeta \neq 0\}|^{1+\varepsilon} . \tag{3.4.13}$$

where $\varepsilon = \min\left\{\frac{2}{n}, \frac{4}{n} - \frac{2}{\ell}, \frac{4}{n} - \frac{2}{p}, \frac{2}{n} - \frac{2}{s}\right\} > 0$, $F = ||f||_{L^\ell(\Omega_1)}$, and

$$C_1 = c_n^2\left[\left(\frac{32\Lambda}{\lambda}\right)^2 + \frac{128\Lambda}{\lambda} + \frac{9\lambda^2}{2} + \left(\frac{32c_n}{\lambda}\right)^2 + \left(\frac{32c_n}{\lambda} ||V_-||_{L^p(\Omega_1)}\right)^2 + 2\left(\frac{32}{\lambda} ||b||_{L^s(\Omega_1)}\right)^2\right].$$

For fixed $0 < r \leq \rho \leq 1$, let $\zeta \in C_c^{\infty}(B_\rho)$ be such that $\zeta \equiv 1$ in B_r and $|D\zeta| \leq \frac{2}{\rho - r}$ in B_1. We let $A(k,r) = \{x \in \Omega_r : u \geq k\} = \operatorname{supp} v \cap \Omega_r$. Then, for any $0 < r < \rho \leq 1$ and $k \geq 0$, if (3.4.12) holds, then (3.4.13) implies that

$$\int_{A(k,r)} (u-k)^2 \leq C_1 \left[\frac{|A(k,\rho)|^{\varepsilon}}{(\rho - r)^2} \int_{A(k,\rho)} (u-k)^2 + (F+k)^2 |A(k,\rho)|^{1+\varepsilon} \right]. \quad (3.4.14)$$

Since the Hölder inequality implies that

$$|A(k,r)| \leq \frac{1}{k} \int_{A(k,r)} u^+ \leq \frac{1}{k} \left(\int_{\Omega_R} |u^+|^2 \right)^{\frac{1}{2}} |A(k,r)|^{\frac{1}{2}},$$

then $|\{v\zeta \neq 0\}| \leq \dfrac{1}{k^2} \left|\left|u^+\right|\right|_{L^2(\Omega_R)}^2$. To ensure that (3.4.12) holds, we take

$$k \geq k_0 := \mathcal{C} \left|\left|u^+\right|\right|_{L^2(\Omega_R)}, \quad (3.4.15)$$

where

$$\mathcal{C} := \left(\frac{32 c_n}{\lambda} ||b||_{L^s(\Omega_1)} \right)^{\frac{ns}{2s-2n}} + \left(\frac{32 c_n}{\lambda} ||d||_{L^t(\Omega_1)} \right)^{\frac{nt}{2t-2n}} + \left(\frac{32 c_n^2}{\lambda} ||V_-||_{L^p(\Omega_1)} \right)^{\frac{np}{4p-2n}}.$$

The goal is to show that there exists a $k \geq k_0$ such that

$$\int_{A(k,1/2)} (u-k)^2 = 0.$$

With $h > k \geq k_0$ and $0 < r < 1$, it follows from the arguments in the previous proof that

$$\left|\left|(u-h)^+\right|\right|_{L^2(\Omega_r)} \leq C_2 \left[\frac{1}{(\rho - r)(h-k)^{\varepsilon}} + \frac{F+h}{(h-k)^{1+\varepsilon}} \right] \left|\left|(u-k)^+\right|\right|_{L^2(\Omega_\rho)}^{1+\varepsilon}, \quad (3.4.16)$$

where C_2 depends on C_1.

Set $\varphi(k,r) = \left|\left|(u-k)^+\right|\right|_{L^2(\Omega_r)}$. For $i = 0, 1, 2, \ldots$, define

$$k_i = k_0 + K\left(1 - \frac{1}{2^i}\right), \qquad r_i = \frac{1}{2} + \frac{1}{2^{i+1}},$$

where $K > 0$ is to be determined. Then it follows from (3.4.16) with $\rho = r_{i-1}$, $r = r_i$, $h = k_i$, and $k = k_{i-1}$ that for $i \geq 1$

$$\varphi(k_i, r_i) \leq C_2 \left[3 \frac{2^{(1+\varepsilon)i}}{K^{\varepsilon}} + (F + k_0) \left(\frac{2^i}{K} \right)^{1+\varepsilon} \right] \varphi(k_{i-1}, r_{i-1})^{1+\varepsilon}. \quad (3.4.17)$$

Claim: There exists $\mu > 1$ and K sufficiently large (depending, in particular, on μ) such that for any $i = 0, 1, \ldots$ (3.4.10) holds.

Clearly, the claim holds for $i = 0$. If the claim holds for $i - 1$, then

$$\varphi(k_i, r_i) \le C_2 \mu^{1+\varepsilon} \left[3 + \frac{F + k_0}{K}\right] \left(\frac{2^{1+\varepsilon}}{\mu^\varepsilon}\right)^i \left(\frac{\varphi(k_0, r_0)}{K}\right)^\varepsilon \frac{\varphi(k_0, r_0)}{\mu^i}.$$

If we choose $\mu > 1$ so that $\mu^\varepsilon \ge 2^{1+\varepsilon}$, then for the claim to hold we need

$$C_2 \mu^{1+\varepsilon} \left[3 + \frac{F + k_0}{K}\right] \left(\frac{\varphi(k_0, r_0)}{K}\right)^\varepsilon \le 1.$$

Setting $K = C_0 \varphi(k_0, r_0) + F + k_0$ for some $C_0 >> 1$ that depends on C_2, μ and ε, gives the claim.

Taking $i \to \infty$ in (3.4.10) shows that $\varphi\left(k_0 + K, \frac{1}{2}\right) = 0$. Since $\varphi(k_0, r_0) = \varphi(k_0, 1) \le ||u^+||_{L^2(\Omega_1)}$, then

$$\sup_{\Omega_{1/2}} u^+ \le K + k_0 \le C_0 \left|\left|u^+\right|\right|_{L^2(\Omega_R)} + F + 2k_0 = C_3 \left|\left|u^+\right|\right|_{L^2(\Omega_1)} + ||f||_{L^\ell(\Omega_1)},$$

where $C_3 = C_0 + 2\mathcal{C}$.

The estimate for $R \neq 1$ follows from a standard scaling argument. Assume that $\mathcal{L}u = f$ weakly on Ω_R. Let $u_R(x) = u(Rx)$, $V_R(x) = R^2 V(Rx)$, $b_R(x) = Rb(Rx)$, $d_R(x) = Rd(Rx)$, $f_R(x) = R^2 f(Rx)$, and define $\mathcal{L}_R$ to be the scaled version of $\mathcal{L}$. Then $\mathcal{L}_R u_R = f_R$ on B_1. Since $\mathcal{L}_R$ has the same ellipticity constant as $\mathcal{L}$, then by the previous estimate,

$$\sup_{\Omega_{R/2}} u^+ = \sup_{\Omega_{1/2}} u_R^+ \le C_{3,R} \left|\left|u_R^+\right|\right|_{L^2(\Omega_1)} + ||f_R||_{L^\ell(\Omega_1)} \le C_{3,R} R^{-n/2} \left|\left|u^+\right|\right|_{L^2(\Omega_R)} + R^{2-\frac{n}{\ell}} ||f||_{L^\ell(\Omega_R)},$$

where

$$\begin{aligned} C_{3,R} = c\Bigg[&\left(\frac{32\Lambda}{\lambda}\right)^2 + \frac{128\Lambda}{\lambda} + \frac{9\lambda^2}{2} + \left(\frac{32c_n}{\lambda}\right)^2 + \left(\frac{32c_n}{\lambda} R^{2-\frac{n}{p}} ||V_-||_{L^p(\Omega_R)}\right)^2 \\ &+ 2\left(\frac{32}{\lambda} R^{1-\frac{n}{s}} ||b||_{L^s(\Omega_R)}\right)^2 \Bigg]^{c_1} + 2\Bigg[\left(\frac{32c_n^2}{\lambda} R^{2-\frac{n}{p}} ||V_-||_{L^p(\Omega_R)}\right)^{\frac{np}{4p-2n}} \\ &+ \left(\frac{32c_n}{\lambda} R^{1-\frac{n}{s}} ||b||_{L^s(\Omega_R)}\right)^{\frac{ns}{2s-2n}} + \left(\frac{32c_n}{\lambda} R^{1-\frac{n}{t}} ||d||_{L^t(\Omega_R)}\right)^{\frac{nt}{2t-2n}} \Bigg] \end{aligned}$$

grows with R.

The rest of the proof, which includes $q \neq 2$ and $r = \theta R$ for $\theta \neq \frac{1}{2}$, follows that of the previous lemma. $\square$

3.5 Interior Hölder continuity in the equation setting

Within this section, we prove Hölder continuity of solutions to general second-order elliptic equations with lower order terms. Towards proving Hölder continuity of solutions, we first show that a lower bound holds for all non-negative supersolutions to our PDE. The combination of this lower bound with the upper bounds in Section 3.4 and the arguments presented in Corollary 4.18 from [24] leads to the proof of Hölder continuity.

To prove the lower bound, we use some of the ideas presented in [24], but since lower order terms were not considered there, we have added the details. Again, the general approach that we follow is based on the ideas of de Giorgi. Similar estimates are presented in [22] using Moser's approach. We actually avoid the use of Moser's iteration, and as a consequence, we prove a lower bound for u in terms of $||u||_{q_0}$ for only one q_0 instead of a full range of values as was done in [24] and [22]. For us, the lower bound is a step towards Hölder continuity, so a single q_0 is sufficient.

Since our proofs are different from those in [24] and [22], we have included the details here. We also present the structure of the associated constants.

To start, we prove the following result that uses the John-Nirenberg lemma.

Lemma 3.5.1. *Take* $N = 1$. *Assume that* $\mathbf{F}(B_R)$, $\mathbf{F}_0(B_R)$, $\mathcal{L}$, *and* $\mathcal{B}$ *satisfy* A1) – A5) and B2). Suppose $V = V_+ - V_-$ where $V_\pm \geq 0$ a.e. and $V_+ \in L^p(B_R)$ for some $p \in \left(\frac{n}{2}, \infty\right]$. Assume that there exists $s, t \in (n, \infty]$ so that $b \in L^s(B_R)^n$, $d \in L^t(B_R)^n$. Assume that $f \in L^\ell(B_R)$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$, $g^\alpha \in L^m(B_R)$ for some $m \in (n, \infty]$. Let $u \in \mathbf{F}(B_R)$ be a non-negative supersolution in the sense that for any $\varphi \in \mathbf{F}_0(B_R)$ such that $\varphi \geq 0$ in B_R, we have

$$\mathcal{B}[u, \varphi] \geq -\int f\varphi + \int g^\alpha D_\alpha \varphi. \tag{3.5.1}$$

Then there exists $q_0\left(n, p, s, t, \lambda, \Lambda, R^{2-\frac{n}{p}}||V_+||_{L^p(B_R)}, R^{1-\frac{n}{s}}||b||_{L^s(B_R)}, R^{1-\frac{n}{t}}||d||_{L^t(B_R)}\right) > 0$ so that for any $k \geq |B_R|^{\frac{2}{n}-\frac{1}{\ell}}||f||_{L^\ell(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{m}}||g||_{L^m(B_R)}$, and any $B_r(y) \subset B_{3R/4}$,

$$\int_{B_r(y)} (u+k)^{-q_0} \int_{B_r(y)} (u+k)^{q_0} \leq C_n r^{2n}. \tag{3.5.2}$$

Remark 3.5.2. This lemma is analogous to the first step of the proof of Theorem 4.15 from [24], except that here we have lower order terms.

Proof. Let $\zeta \in C_c^\infty(B_R)$ be a cutoff function, $0 \le \zeta \le 1$. By B2) with $\omega = 1$, for any $k > 0$, $\bar{u} := (u+k)^{-1} \in \mathbf{F}(B_R)$. It follows from A4) that $\varphi := \bar{u}\zeta^2 \in \mathbf{F}_0(B_R)$. Since u is a supersolution, we have

$$\begin{aligned}
0 &\le \int \left(A^{\alpha\beta} D_\beta u + b^\alpha u\right) D_\alpha \varphi + d^\beta D_\beta u \varphi + V u \varphi + \int f \bar{u}^{-1} \zeta^2 - \int g^\alpha D_\alpha \varphi \\
&= -\int A^{\alpha\beta}\, D_\beta w\, D_\alpha w\, \zeta^2 + 2\int A^{\alpha\beta}\, D_\beta w\, D_\alpha \zeta\, \zeta - \int \left(1 - \frac{k}{\bar{u}}\right) b^\alpha D_\alpha w\, \zeta^2 \\
&\quad + 2\int \left(1 - \frac{k}{\bar{u}}\right) b^\alpha D_\alpha \zeta\, \zeta + \int d^\beta D_\beta w\, \zeta^2 + \int V \left(1 - \frac{k}{\bar{u}}\right) \zeta^2 + \int \frac{f}{\bar{u}} \zeta^2 \\
&\quad + \int \frac{g^\alpha}{\bar{u}} D_\alpha w \zeta^2 - 2\int \frac{g^\alpha}{\bar{u}} \zeta D_\alpha \zeta,
\end{aligned}$$

where we have set $w = \log \bar{u}$. With $\tilde{f} := \dfrac{f}{\bar{u}}$, $\tilde{g} := \dfrac{|g|}{\bar{u}}$, we rearrange and bound to get

$$\begin{aligned}
\lambda \int |Dw|^2 \zeta^2 &\le \int A^{\alpha\beta}\, D_\beta w\, D_\alpha w\, \zeta^2 \\
&\le 2\Lambda \int |Dw|\ |D\zeta|\ \zeta + \int \left(|b| + |d| + |\tilde{g}|\right) |Dw|\ \zeta^2 + 2\int \left(|b| + |\tilde{g}|\right) |D\zeta|\ \zeta \\
&\qquad + \int \left(|V_+| + \tilde{f}\right) \zeta^2 \\
&\le \frac{\lambda}{2} \int |Dw|^2 \zeta^2 + C_1 \int |D\zeta|^2,
\end{aligned}$$

where

$$\begin{aligned}
C_1 = \frac{8\Lambda^2}{\lambda} &+ \frac{2c_n^2}{\lambda} \left(|B_R|^{\frac{1}{n}-\frac{1}{s}} ||b||_{L^s(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{t}} ||d||_{L^t(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{m}} ||\tilde{g}||_{L^m(B_R)}\right)^2 \\
&+ 2c_n \left(|B_R|^{\frac{1}{n}-\frac{1}{s}} ||b||_{L^s(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{m}} ||\tilde{g}||_{L^m(B_R)}\right) \\
&+ c_n^2 \left(|B_R|^{\frac{2}{n}-\frac{1}{p}} ||V_+||_{L^p(B_R)} + |B_R|^{\frac{2}{n}-\frac{1}{\ell}} \left|\left|\tilde{f}\right|\right|_{L^\ell(B_R)}\right).
\end{aligned}$$

If $|B_R|^{\frac{2}{n}-\frac{1}{\ell}} ||f||_{L^\ell(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{m}} ||g||_{L^m(B_R)} > 0$, then we choose $k = |B_R|^{\frac{2}{n}-\frac{1}{\ell}} ||f||_{L^\ell(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{m}} ||g||_{L^m(B_R)}$. Otherwise, we choose $k > 0$ to be arbitrary and eventually take $k \to 0^+$. Then

$$\int |Dw|^2 \zeta^2 \le C_2 \int |D\zeta|^2, \tag{3.5.3}$$

where

$$C_2 = \left(\frac{4\Lambda}{\lambda}\right)^2 + \left(\frac{2c_n}{\lambda}\right)^2 \left(|B_R|^{\frac{1}{n}-\frac{1}{s}} \|b\|_{L^s(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{t}} \|d\|_{L^t(B_R)} + 1\right)^2$$
$$+ \frac{4c_n}{\lambda}\left(|B_R|^{\frac{1}{n}-\frac{1}{s}} \|b\|_{L^s(B_R)} + 1\right) + \frac{2c_n^2}{\lambda}\left(|B_R|^{\frac{2}{n}-\frac{1}{p}} \|V_+\|_{L^p(B_R)} + 1\right). \qquad (3.5.4)$$

Let $B_r(y) \subset B_{3R/4}$. Choose ζ so that $\zeta \equiv 1$ in $B_r(y)$, $\operatorname{supp}\zeta \Subset B_R$, and $|D\zeta| \leq \frac{C}{r}$. It follows from the Hölder inequality, Poincaré inequality, then (3.5.3), that for any $B_r(y) \subset B_{3R/4}$,

$$\int_{B_r(y)} |w - w_{y,r}| \leq |B_r|^{\frac{1}{2}} \left(\int_{B_r(y)} |w - w_{y,r}|^2\right)^{\frac{1}{2}} \leq c_n r^{\frac{n+2}{2}} \left(\int_{B_r} |Dw|^2\right)^{\frac{1}{2}}$$
$$\leq c_n r^{\frac{n+2}{2}} \left(C_2 \int |D\zeta|^2\right)^{\frac{1}{2}} \leq C_3 r^n,$$

where $w_{y,r} = \fint_{B_r(y)} w$ and $C_3 = c_n\sqrt{C_2}$. Therefore, w is a BMO function. By the John-Nirenberg lemma, there exists $q_1, C_4 > 0$, depending only on n, so that for any $B_r(y) \subset B_{3R/4}$

$$\int_{B_r(y)} e^{\frac{q_1}{C_3}|w - w_{y,r}|} \leq C_4 r^n.$$

Therefore, with $q_0 = \frac{q_1}{C_3} = \frac{q_1}{c_n\sqrt{C_2}}$,

$$\int_{B_r(y)} \bar{u}^{-q_0} \int_{B_r(y)} \bar{u}^{q_0} = \int_{B_r(y)} e^{-q_0 \log \bar{u}} \int_{B_r(y)} e^{q_0 \log \bar{u}} = \int_{B_r(y)} e^{-q_0(w - w_{y,r})} \int_{B_r(y)} e^{q_0(w - w_{y,r})}$$
$$= \int_{B_r(y)} e^{q_0|w - w_{y,r}|} \int_{B_r(y)} e^{-q_0|w - w_{y,r}|} \leq C_4 r^{2n}.$$

$\square$

Remark 3.5.3. We sometimes use the notation $q_0(R)$ to refer to the exponent q_0 associated to the ball of radius R.

With the previous estimate, we can prove a lower bound for solutions.

Lemma 3.5.4. *Take* $N = 1$. *Assume that* $\mathbf{F}(B_R)$, $\mathbf{F}_0(B_R)$, $\mathcal{L}$, *and* $\mathcal{B}$ *satisfy* A1) – A5) and B1) – B2). Assume that there exists $p \in \left(\frac{n}{2}, \infty\right]$, $s, t \in (n, \infty]$ so that $V_+ \in L^p(B_R)$, $b \in L^s(B_R)$, $d \in L^t(B_R)$. Assume that $f \in L^\ell(B_R)$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$, $g^\alpha \in L^m(B_R)$

for some $m \in (n, \infty]$. Suppose $u \in \mathbf{F}(B_R)$ is a nonnegative supersolution in the sense that for any $\varphi \in \mathbf{F}_0(B_R)$ such that $\varphi \geq 0$ in B_R, (3.5.1) holds. Then for $q_0 = q_0(R)$ (see Remark 3.5.3), we have

$$\left(\fint_{B_{3R/4}} u^{q_0}\right)^{\frac{1}{q_0}} \leq C_0 \left(\inf_{B_{R/2}} u + |B_R|^{\frac{2}{n}-\frac{1}{\ell}} ||f||_{L^\ell(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{m}} ||g||_{L^m(B_R)}\right),$$

where $C_0 = C_0\left(n, q_0, p, s, t, \ell, m, \lambda, \Lambda, R^{2-\frac{n}{p}} ||V_+||_{L^p(B_R)}, R^{1-\frac{n}{s}} ||b||_{L^s(B_R)}, R^{1-\frac{n}{t}} ||d||_{L^s(B_R)}\right)$.

Proof. If $|B_R|^{\frac{2}{n}-\frac{1}{\ell}} ||f||_{L^\ell(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{m}} ||g||_{L^m(B_R)} > 0$, let $k = |B_R|^{\frac{2}{n}-\frac{1}{\ell}} ||f||_{L^\ell(B_R)} + |B_R|^{\frac{1}{n}-\frac{1}{m}} ||g||_{L^m(B_R)}$. Otherwise, if $f, g \equiv 0$, let $k > 0$ and eventually take to $k \to 0^+$. Set $\bar{u} = u + k$. Let $\xi \in C_c^\infty(B_R)$, $\xi \geq 0$, and set $\varphi = \bar{u}^{-\left(1+\frac{1}{2}q_0\right)}\xi \geq 0$, where $q_0 = q_0(R)$ is the constant given to us in Lemma 3.5.1. By B2) with $\omega = 1 + \frac{q_0}{2}$ and an application of A4), $\varphi \in \mathbf{F}_0(B_R)$, so we may use it as a test function.

Set $w = \bar{u}^{-\frac{q_0}{2}}$ so that $Dw = -\frac{q_0}{2}\bar{u}^{-\left(1+\frac{q_0}{2}\right)}D\bar{u}$. By B2), $w \in \mathbf{F}(B_R)$ as well. Then

$$\begin{aligned}
&\int A^{\alpha\beta} D_\beta u\, D_\alpha\varphi + b^\alpha\, u\, D_\alpha\varphi + d^\beta\, D_\beta u\, \varphi + V\, u\, \varphi \\
&= -\frac{2}{q_0}\int A^{\alpha\beta} D_\beta w\, D_\alpha\xi - 4\frac{4+2q_0}{q_0^2}\int A^{\alpha\beta} D_\beta\left(\bar{u}^{-\frac{1}{4}q_0}\right) D_\alpha\left(\bar{u}^{-\frac{1}{4}q_0}\right)\xi + \int\left(1-\frac{k}{\bar{u}}\right) b^\alpha\, w\, D_\alpha\xi \\
&\quad + \frac{2+q_0}{q_0}\int\left(1-\frac{k}{\bar{u}}\right) b^\alpha\, D_\alpha w\xi - \frac{2}{q_0}\int d^\beta\, D_\beta w\, \xi + \int\left(1-\frac{k}{\bar{u}}\right) V\, w\, \xi.
\end{aligned}$$

It follows from (3.5.1), with $\tilde{f} = \dfrac{f}{\bar{u}}$, $\tilde{g}^\alpha = \dfrac{g^\alpha}{\bar{u}}$ that

$$\begin{aligned}
\int A^{\alpha\beta} D_\beta w\, D_\alpha\xi \leq & -4\frac{2+q_0}{q_0}\int A^{\alpha\beta} D_\beta\left(\bar{u}^{-\frac{q_0}{4}}\right) D_\alpha\left(\bar{u}^{-\frac{q_0}{4}}\right)\xi + \frac{q_0}{2}\int\left(1-\frac{k}{\bar{u}}\right) b^\alpha\, w\, D_\alpha\xi \\
& + \left(1+\frac{q_0}{2}\right)\int\left(1-\frac{k}{\bar{u}}\right) b^\alpha\, D_\alpha w\xi - \int d^\beta\, D_\beta w\, \xi + \frac{q_0}{2}\int\left(1-\frac{k}{\bar{u}}\right) V\, w\, \xi \\
& + \frac{q_0}{2}\int \tilde{f}\, w\, \xi - \left(1+\frac{q_0}{2}\right)\int \tilde{g}^\alpha D_\alpha w\, \xi - \frac{q_0}{2}\int \tilde{g}^\alpha w\, D_\alpha\xi.
\end{aligned}$$

Therefore, with $\tilde{b}^\alpha = \frac{q_0}{2}\left[\tilde{g}^\alpha + \left(\frac{k}{\bar{u}} - 1\right) b^\alpha\right]$, $\tilde{d}^\beta = d^\beta + \left(1+\frac{q_0}{2}\right)\left[\left(\frac{k}{\bar{u}} - 1\right) b^\beta + \tilde{g}^\beta\right]$, and $\tilde{V} = -\frac{q_0}{2}\left(V_+ + \tilde{f}\right)$, we have that

$$\begin{aligned}
&\int A^{\alpha\beta} D_\beta w\, D_\alpha\xi + \tilde{b}^\alpha w\, D_\alpha\xi + \tilde{d}^\beta D_\beta w\, \xi - \tilde{V}\, w\, \xi \\
&\qquad \leq -4\left(1+\frac{2}{q_0}\right)\int A^{\alpha\beta} D_\beta\left(\bar{u}^{-\frac{q_0}{4}}\right) D_\alpha\left(\bar{u}^{-\frac{q_0}{4}}\right)\xi \leq 0.
\end{aligned}$$

Since $\xi \in C_c^\infty(B_R)$ is arbitrary and nonnegative, then it follows from A2) that $\widetilde{\mathcal{L}}w \le 0$ in B_R in the weak sense. We may apply Lemma 3.4.6 to w. Thus,

$$\sup_{B_{R/2}} w \le CR^{-\frac{n}{2}} ||w||_{L^2\left(B_{3R/4}\right)},$$

where $C = C\left(n, q_0, p, s, t, \ell, m, \lambda, \Lambda, R^{2-\frac{n}{p}} ||V_+||_{L^p(\Omega_R)}, R^{1-\frac{n}{s}} ||b||_{L^s(\Omega_R)}, R^{1-\frac{n}{t}} ||d||_{L^t(\Omega_R)}\right)$. Since $w = \bar{u}^{-\frac{1}{2}q_0}$ and $\bar{u} = u + k$, then

$$\inf_{B_{R/2}} u + k = \inf_{B_{R/2}} \bar{u} = \left(\sup_{B_{R/2}} w\right)^{-\frac{2}{q_0}} \ge \left(CR^{-\frac{n}{2}} ||w||_{L^2\left(B_{3R/4}\right)}\right)^{-\frac{2}{q_0}} \ge C^{-\frac{2}{q_0}} R^{\frac{n}{q_0}} \left(\int_{B_{3R/4}} \bar{u}^{-q_0}\right)^{-\frac{1}{q_0}}.$$

By Lemma 3.5.1,

$$\left(\int_{B_{3R/4}} \bar{u}^{-q_0}\right)^{-\frac{1}{q_0}} \ge \left[C_n R^n \left(\fint_{B_{3R/4}} \bar{u}^{q_0}\right)^{-1}\right]^{-\frac{1}{q_0}}$$

and therefore,

$$\inf_{B_{R/2}} u + k \ge \left(C^2 C_n\right)^{-\frac{1}{q_0}} \left(\fint_{B_{3R/4}} \bar{u}^{q_0}\right)^{\frac{1}{q_0}} \ge \left(C^2 C_n\right)^{-\frac{1}{q_0}} \left(\fint_{B_{3R/4}} u^{q_0}\right)^{\frac{1}{q_0}},$$

since $\bar{u} \ge u \ge 0$. □

By combining our upper and lower bounds, we arrive at the following Harnack inequality.

Lemma 3.5.5. *Take $N = 1$. Assume that $\mathbf{F}(B_{2R})$, $\mathbf{F}_0(B_{2R})$, $\mathcal{L}$, and $\mathcal{B}$ satisfy* A1) – A5) *and* B1) – B2). *Assume that there exists $p \in \left(\frac{n}{2}, \infty\right]$, $s, t \in (n, \infty]$ so that $V \in L^p(B_R)$, $b \in L^s(B_R)^n$, and $d \in L^t(B_R)^n$. Let $f \in L^\ell(B_R)$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$. Let $u \in \mathbf{F}(B_{2R})$ be a non-negative solution in the sense that $\mathcal{B}[u, \varphi] = \displaystyle\int f\varphi$ for any $\varphi \in \mathbf{F}_0(B_R)$. Then*

$$\sup_{B_{R/4}} u \le C(R) \inf_{B_{R/2}} u + c(R) R^{2-\frac{n}{\ell}} ||f||_{L^\ell(B_R)},$$

where $C(R) = CC_0 \left|B_{3/4}\right|^{\frac{1}{q_0}}$ and $c(R) = CC_0 \left|B_{3/4}\right|^{\frac{1}{q_0}} |B_1|^{\frac{2}{n}-\frac{1}{\ell}} + c_{q_0}$, with $q_0 = q_0(R)$, C and c_{q_0} as in Lemma 3.4.6, and C_0 as in Lemma 3.5.4.

The proof is an application of Lemmas 3.4.6 and 3.5.4.

Proof. By Lemma 3.4.6 with $q_0 = q_0(R)$,

$$\sup_{B_{R/4}} u \le C R^{-\frac{n}{q_0}} ||u||_{L^{q_0}\left(B_{3R/4}\right)} + c_{q_0} R^{2-\frac{n}{\ell}} ||f||_{L^\ell(B_R)},$$

where $C = C\left(n, q_0, p, s, t, \ell, \lambda, \Lambda, R^{2-\frac{n}{p}} ||V_-||_{L^p(\Omega_R)}, R^{1-\frac{n}{s}} ||b||_{L^s(\Omega_R)}, R^{1-\frac{n}{t}} ||d||_{L^t(\Omega_R)}\right)$.
By Lemma 3.5.4,

$$\left(\fint_{B_{3R/4}} u^{q_0}\right)^{\frac{1}{q_0}} \le C_0 \left(\inf_{B_{R/2}} u + |B_R|^{\frac{2}{n}-\frac{1}{\ell}} ||f||_{L^\ell(B_R)}\right),$$

where $C_0 = C_0\left(n, q_0, p, s, t, \ell, \lambda, \Lambda, R^{2-\frac{n}{p}} ||V_+||_{L^p(B_R)}, R^{1-\frac{n}{s}} ||b||_{L^s(B_R)}, R^{1-\frac{n}{t}} ||d||_{L^s(B_R)}\right)$.
Thus,

$$\sup_{B_{R/4}} u \le C C_0 \left|B_{3/4}\right|^{\frac{1}{q_0}} \inf_{B_{R/2}} u + \left(C C_0 \left|B_{3/4}\right|^{\frac{1}{q_0}} |B_1|^{\frac{2}{n}-\frac{1}{\ell}} + c_{q_0}\right) R^{2-\frac{n}{\ell}} ||f||_{L^\ell(B_R)}$$

$\square$

Now we have all of the tools we need to prove interior Hölder continuity of solutions.

Lemma 3.5.6. *Take* $N = 1$. *Assume that* $\mathbf{F}(B_{2R_0})$, $\mathbf{F}_0(B_{2R_0})$, $\mathcal{L}$, *and* $\mathcal{B}$ *satisfy* A1) – A5) and B1) – B2). Assume that there exists $p \in \left(\frac{n}{2}, \infty\right]$, $s, t \in (n, \infty]$ so that $V \in L^p(B_{R_0})$, $b \in L^s(B_{R_0})^n$, and $d \in L^t(B_{R_0})^n$. Let $u \in \mathbf{F}(B_{2R_0})$ be a solution in the sense that $\mathcal{B}[u, \varphi] = 0$ for any $\varphi \in \mathbf{F}_0(B_{R_0})$. Let $C_0 = C_0(R_0)$ be as given in Lemma 3.5.4. Then there exists $\eta(n, p, s, C_0) \in (0, 1)$, such that for any $R \le R_0$, if $x, y \in B_{R/2}$

$$|u(x) - u(y)| \le C \left(\frac{|x-y|}{R}\right)^\eta \left(\fint_{B_R} |u|^{2^*}\right)^{\frac{1}{2^*}},$$

where $C\left(n, p, s, t, \lambda, \Lambda, \eta, C_0(R_0), R_0^{2-\frac{n}{p}} ||V||_{L^p\left(B_{R_0}\right)}, R_0^{1-\frac{n}{s}} ||b||_{L^s\left(B_{R_0}\right)}, R_0^{1-\frac{n}{t}} ||d||_{L^t\left(B_{R_0}\right)}\right)$.

Proof. Assume first that $R = 2$. For $r \in (0, 1)$, let $m(r) = \inf_{B_r} u$, $M(r) = \sup_{B_r} u$. By our previous results, $-\infty < m(r) \le M(r) < \infty$. Set $M_0 = \sup_{B_1} |u|$. Let $q_0 = q_0(1)$ as given in Lemma 3.5.1. The Minkowski inequality shows that

$$M(r)-m(r)=\left(\fint_{B_{3r/4}}|M(r)-m(r)|^{q_0}\right)^{\frac{1}{q_0}}$$
$$\leq\left(\fint_{B_{3r/4}}|M(r)-u|^{q_0}\right)^{\frac{1}{q_0}}+\left(\fint_{B_{3r/4}}|u-m(r)|^{q_0}\right)^{\frac{1}{q_0}}. \tag{3.5.5}$$

Let $\varphi\in\mathbf{F}_0(B_r)$ be such that $\varphi\geq 0$ in B_r. Since $M(r)-u\geq 0$ and

$$\begin{aligned}\mathcal{B}[M(r)-u,\varphi]&=\int\left[A^{\alpha\beta}D_\beta(M(r)-u)+b^\alpha(M(r)-u)\right]D_\alpha\varphi\\&\quad+\left[d^\beta D_\beta(M(r)-u)+V(M(r)-u)\right]\varphi\\&=-\mathcal{B}[u,\varphi]+M(r)\int(V-D_\alpha b^\alpha)\varphi=M(r)\int(V-D_\alpha b^\alpha)\varphi,\end{aligned}$$

then by Lemma 3.5.4 with $f:=-M(r)V\in L^p(B_r)$ and $g^\alpha:=M(r)b^\alpha\in L^s(B_r)$,

$$\begin{aligned}&\left(\fint_{B_{3r/4}}|M(r)-u|^{q_0}\right)^{\frac{1}{q_0}}\\&\quad\leq C_0\left[\inf_{B_{r/2}}[M(r)-u]+M_0\left(|B_r|^{\frac{2}{n}-\frac{1}{p}}||V||_{L^p(B_r)}+|B_r|^{\frac{1}{n}-\frac{1}{s}}||b||_{L^s(B_r)}\right)\right].\end{aligned} \tag{3.5.6}$$

Similarly, since $u-m(r)\geq 0$ and

$$\begin{aligned}\mathcal{B}[u-m(r),\varphi]&=\int\left[A^{\alpha\beta}D_\beta(u-m(r))+b^\alpha(u-m(r))\right]D_\alpha\varphi\\&\quad+\left[d^\beta D_\beta(u-m(r))+V(u-m(r))\right]\varphi\\&=\mathcal{B}[u,\varphi]-m(r)\int(V-D_\alpha b^\alpha)\varphi=-m(r)\int(V-D_\alpha b^\alpha)\varphi,\end{aligned}$$

then

$$\begin{aligned}&\left(\fint_{B_{3r/4}}|u-m(r)|^{q_0}\right)^{\frac{1}{q_0}}\\&\quad\leq C_0\left[\inf_{B_{r/2}}[u-m(r)]+M_0\left(|B_r|^{\frac{2}{n}-\frac{1}{p}}||V||_{L^p(B_r)}+|B_r|^{\frac{1}{n}-\frac{1}{s}}||b||_{L^s(B_r)}\right)\right].\end{aligned} \tag{3.5.7}$$

Combining (3.5.5), (3.5.6) and (3.5.7), we see that

$$\begin{aligned}\frac{1}{C_0}[M(r)-m(r)]&\leq M(r)-M\left(\frac{r}{2}\right)+m\left(\frac{r}{2}\right)-m(r)\\&\quad+2M_0\left(|B_1|^{\frac{2}{n}-\frac{1}{p}}r^{2-\frac{n}{p}}||V||_{L^p(B_r)}+|B_1|^{\frac{1}{n}-\frac{1}{s}}r^{1-\frac{n}{s}}||b||_{L^s(B_r)}\right).\end{aligned}$$

Set $\omega(r) = \operatorname{osc}_{B_r} u = M(r) - m(r)$, $\delta = \min\left\{2 - \frac{n}{p}, 1 - \frac{n}{s}\right\}$, $c = 2\max\left\{|B_1|^{\frac{2}{n}-\frac{1}{p}}, |B_1|^{\frac{1}{n}-\frac{1}{s}}\right\}$. Since $C_0 = C_0(r)$ is monotonically increasing,

$$\omega\left(\frac{r}{2}\right) \leq \left(1 - \frac{1}{C_0(1)}\right)\omega(r) + cr^{\delta} M_0 \left(||V||_{L^p(B_1)} + ||b||_{L^s(B_1)}\right).$$

Choose $\mu \in (0,1)$, so that $\eta := (1-\mu)\frac{\log\left(1 - C_0(1)^{-1}\right)}{\log\left(\frac{1}{2}\right)} < \mu\delta$. For any such η, it follows from Lemma 4.19 in [24] that for any $\rho \in [0,1)$,

$$\omega(\rho) \leq \frac{2^{\eta}}{1 - C_0(1)^{-1}}\rho^{\eta}\omega(1) + \frac{c\, C_0(1)}{2^{\delta(1-\mu)}}\left(||V||_{L^p(B_1)} + ||b||_{L^s(B_r)}\right)\rho^{\eta} M_0.$$

By Lemma 3.4.6,

$$\omega(1) \leq C\left(\int_{B_2} |u|^{2^*}\right)^{\frac{1}{2^*}}$$

$$M_0 = \sup_{B_1}|u| \leq C\left(\int_{B_2} |u|^{2^*}\right)^{\frac{1}{2^*}}.$$

Thus,

$$\omega(\rho) \leq C\rho^{\eta}\left(\int_{B_2} |u|^{2^*}\right)^{\frac{1}{2^*}},$$

where $C\left(n, p, s, t, \lambda, \Lambda, \eta, C_0(1), ||V||_{L^p(B_2)}, ||b||_{L^s(B_2)}, ||d||_{L^t(B_2)}\right)$. The usual scaling argument gives the general result. $\square$

3.6 Examples

We now show that a number of cases satisfy the assumptions from our general set-up:

Case 1. *Homogeneous operators*: When $\mathbf{b}, \mathbf{d}, \mathbf{V} \equiv \mathbf{0}$, take $\mathbf{F}(\Omega) = Y^{1,2}(\Omega)^N$. This case was studied by Hofmann and Kim in [27] and fits into our framework.

Case 2. *Lower order coefficients in L^p, Sobolev space*: When $\mathbf{b}, \mathbf{d}, \mathbf{V}$ are in some L^p spaces and satisfy a non-degeneracy condition, $\mathbf{F}(\Omega) = W^{1,2}(\Omega)^N$.

Case 3: *Reverse Hölder potentials*: When $\mathbf{V} \in B_p$ for some $p \in \left[\frac{n}{2}, \infty\right)$ (to be defined below), $\mathbf{b}, \mathbf{d} \equiv \mathbf{0}$, we define $\mathbf{F}(\Omega) = W^{1,2}_V(\Omega)^N$, a weighted Sobolev space, with the weight function depending on the potential function $\mathbf{V}$.

The goal of this section is to show that each of the three cases listed above fits into the framework described in the Section 3.1. More specifically, we first show that $\mathbf{F}(\Omega)$ and $\mathbf{F}_0(\Omega)$ satisfy assumptions A1)–A4). Then we show that A5)–A7) hold for $\mathbf{F}(\Omega)$, $\mathbf{F}_0(\Omega)$, $\mathcal{L}$, and $\mathcal{B}$; we prove boundedness as in (3.1.18), coercivity as in (3.1.19), and the Caccioppoli inequality (3.1.20). At this point, if we assume that (IB), (BB), and (H) also hold, then we have the full set of results on fundamental and Green matrices. Going further, we consider the case of real equations (as opposed to real systems), and we justify the assumptions of (IB), (BB), and (H) in each of the cases described above. To this end, due to Sections 3.4 and 3.5, we will only have to show that B1)–B2) hold. We remind the reader that for systems, the assumptions (IB), (BB), and (H) may actually fail.

3.6.1 Homogeneous operators

We start with the case when $\mathbf{V}, \mathbf{b}, \mathbf{d} \equiv \mathbf{0}$, $\mathcal{L} = L$ and

$$\mathcal{B}[\mathbf{u}, \mathbf{v}] = B[\mathbf{u}, \mathbf{v}] := \int \mathbf{A}^{\alpha\beta} D_\beta \mathbf{u} \cdot D_\alpha \mathbf{v} = \int A_{ij}^{\alpha\beta} D_\beta u^j \, D_\alpha v^i.$$

By ellipticity (3.1.9) and boundedness (3.1.10) of the matrix $\mathbf{A}$, $B[\cdot, \cdot]$ is comparable to the inner product given by (2.3.3). Therefore, it is natural to take the Banach space to be $\mathbf{F}(\Omega) = Y^{1,2}(\Omega)^N$, while the associated Hilbert space is $\mathbf{F}_0(\Omega) = Y_0^{1,2}(\Omega)^N$, for all Ω open and connected.

The restriction property (3.1.1) is obviously true in this setting. It is also clear that $C_c^\infty(\Omega)^N$ functions belong to $Y^{1,2}(\Omega)^N$, and, by the discussion in the beginning of Section 3.1, $Y_0^{1,2}(\Omega)^N$ is a Hilbert space equipped with the scalar product (2.3.3). A3) is trivially satisfied.

By Lemma 2.3.27, $C^\infty(U)^N \cap Y^{1,2}(U)^N$ is dense in $Y^{1,2}(U)^N$ for any bounded U. This implies (3.1.4) since we may assume that U in (3.1.4) is bounded because the support of ξ is bounded. With $\xi \in C_c^\infty(U)$, it is immediate that $\mathbf{u}\xi \in L^{2^*}(\Omega \cap U)^N$ and

$$D_\alpha(\mathbf{u}\xi) = \xi D_\alpha \mathbf{u} + \mathbf{u}\mathcal{D}_\alpha \xi \in L^2(\Omega \cap U)^N,$$

where we have used that $\mathbf{u} \in L^{2^*}(\Omega \cap U)^N \hookrightarrow L^2(\Omega \cap U)^N$ since U is bounded. It follows that $||\mathbf{u}\xi||_{Y^{1,2}(\Omega\cap U)} \le C_\xi ||\mathbf{u}||_{Y^{1,2}(\Omega)}$. Now if $\{\mathbf{u}_n\} \subset C^\infty(\Omega \cap U)^N$ approximates

$\mathbf{u}$ in the $Y^{1,2}(\Omega \cap U)^N$-norm, then for $\xi \in C_c^\infty(U)$, we observe that $\{\mathbf{u}_n \xi\} \subset C^\infty(\Omega \cap U)^N$ approximates $\mathbf{u}\xi$ since

$$\|\mathbf{u}_n\xi - \mathbf{u}\xi\|_{Y^{1,2}(\Omega\cap U)} \leq \|D(\mathbf{u}_n - \mathbf{u})\|_{L^2(\Omega\cap U)}\|\xi\|_{L^\infty(\Omega\cap U)} + \|\mathbf{u}_n - \mathbf{u}\|_{L^2(\Omega\cap U)}\|D\xi\|_{L^\infty(\Omega\cap U)} \tag{3.6.1}$$

$$+ \|\mathbf{u}_n - \mathbf{u}\|_{L^{2^*}(\Omega\cap U)}\|\xi\|_{L^\infty(\Omega\cap U)}.$$

Applying the Hölder inequality to the second term, the latter is majorized by $\|\mathbf{u}_n - \mathbf{u}\|_{Y^{1,2}(U\cap\Omega)}$, as desired. A similar argument implies that when $\xi \in C_c^\infty(\Omega \cap U)$, $\{\mathbf{u}_n\xi\} \subset C_c^\infty(\Omega \cap U)^N$ approximates $\mathbf{u}\xi$.

Turning to A5)–A7), (3.1.18) and (3.1.19) follow directly from (3.1.10) and (3.1.9) with $\Gamma = \Lambda$ and $\gamma = \lambda$. The Caccioppoli inequality is well-known in this context, however one can also refer to Lemma 3.3.1. Indeed, since all of the lower order coefficients vanish, then Lemma 3.3.1 applies to give the Caccioppoli inequality (3.1.20) with $C = C(n, \lambda, \Lambda)$. All in all, A1) – A7) are verified in this setting.

Reducing to the case of equations, i.e., $N = 1$, conditions (IB) and (BB) hold with $C = C(n, q, \ell, \lambda, \Lambda)$ due to Lemma 4.1 from [24], or one could also use Lemma 3.4.1 by showing that B1) holds.

If one wants to show B1), it is enough to observe that its proof can be reduced to the case of $\mathbf{F}(\Omega) = W^{1,2}(\Omega)$. This is because $Y_0^{1,2}(\Omega_R) = W_0^{1,2}(\Omega_R)$ by Lemma 2.3.7. Indeed, for any $u \in Y^{1,2}(\Omega_R) \hookrightarrow W^{1,2}(\Omega_R)$ (see Lemma 2.3.7), if $u\zeta \in Y_0^{1,2}(\Omega_R)$ for all $\zeta \in C_c^\infty(B_R)$, then $u\zeta \in W_0^{1,2}(\Omega_R)$. If B1) holds for $\mathbf{F}(\Omega) = W^{1,2}(\Omega)$, we have for all ζ smooth compactly supported non-negative $\zeta(u-k)_+ \in W_0^{1,2}(\Omega_R) = Y_0^{1,2}(\Omega_R)$ by Lemma 2.3.7, as desired. Clearly, the property $D_\alpha\zeta(u-k)_+ \in L^2(\Omega_R)$ is also inherited. We will postpone the proof of B1) for $\mathbf{F}(\Omega) = W^{1,2}(\Omega)$ to Case 2.

In this context, (H) also can be found in the literature, specifically, Corollary 4.18 from [24] applies since the spaces $W^{1,2}(B_R)$ and $Y^{1,2}(B_R)$ coincide for any $B_R \subset \Omega$ (see Corollary 2.3.11). The latter fact also allows us to reduce the proof of B2) to the case of $\mathbf{F}(\Omega) = W^{1,2}(\Omega)$ (discussed below) should we prefer to use Lemma 3.5.6.

3.6.2 Lower order coefficients in L^p, Sobolev space

Assume that there exist exponents $p \in \left(\frac{n}{2}, \infty\right]$, $s, t \in (n, \infty]$ so that $\mathbf{V} \in L^p(\Omega)^{N\times N}$, $\mathbf{b} \in L^s(\Omega)^{n\times N\times N}$, and $\mathbf{d} \in L^t(\Omega)^{n\times N\times N}$. Set $\mathbf{F}(\Omega) = W^{1,2}(\Omega)^N$ and $\mathbf{F}_0(\Omega) = W_0^{1,2}(\Omega)^N$.

To establish the assumptions A1) through A4), we rely on a number of facts regarding Sobolev spaces which are contained in Appendix 2.3.4, with further details in [17], for example.

The property (3.1.1) is straightforward and therefore A1) holds. Clearly, $C_c^\infty(\Omega)^N$ is contained in $W^{1,2}(\Omega)^N$ and the completion, $W_0^{1,2}(\Omega)^N$, is a Hilbert space with respect to $\|\cdot\|_{W_0^{1,2}(\Omega)^N} = \|\cdot\|_{W^{1,2}(\Omega)^N}$. A3) follows from Lemma 2.3.1. For $\mathbf{u} \in W^{1,2}(\Omega)^N$ and $\xi \in C_c^\infty(U)$, boundedness of ξ and $D\xi$ implies that $\mathbf{u}\xi \in W^{1,2}(\Omega \cap U)^N$, and, as in the previous case, $||\mathbf{u}\xi||_{W^{1,2}(\Omega\cap U)} \le C_\xi ||\mathbf{u}||_{W^{1,2}(\Omega)}$. By Lemma 2.3.27, $C^\infty(U)^N \cap W^{1,2}(U)^N$ is dense in $W^{1,2}(U)^N$, so that (3.1.4), and hence A4), holds by the same argument as in Case 1, similar to (3.6.1).

Boundedness of the matrix $\mathbf{A}$, (3.1.10), implies that for any $\mathbf{u}, \mathbf{v} \in W_0^{1,2}(\Omega)^N$,

$$\mathcal{B}[\mathbf{u},\mathbf{v}] \le \Lambda \int |D\mathbf{u}|\,|D\mathbf{v}| + \int |\mathbf{b}|\,|\mathbf{u}|\,|D\mathbf{v}| + \int |\mathbf{d}|\,|D\mathbf{u}|\,|\mathbf{v}| + \int |\mathbf{V}|\,|\mathbf{u}|\,|\mathbf{v}|.$$

By the Hölder inequality

$$\int |D\mathbf{u}|\,|D\mathbf{v}| \le \left(\int |D\mathbf{u}|^2\right)^{\frac{1}{2}} \left(\int |D\mathbf{v}|^2\right)^{\frac{1}{2}}$$

By Hölder, homogeneous Sobolev and Young's inequalities, since $s \in (n, \infty]$,

$$\begin{aligned}
\int |\mathbf{b}|\,|\mathbf{u}|\,|D\mathbf{v}| &= \int |\mathbf{b}|\,|\mathbf{u}|^{\frac{s-n}{s}}\,|\mathbf{u}|^{\frac{n}{s}}\,|D\mathbf{v}| \le \left(\int |\mathbf{b}|^s\right)^{\frac{1}{s}} \left(\int |\mathbf{u}|^2\right)^{\frac{s-n}{2s}} \left(\int |\mathbf{u}|^{2^*}\right)^{\frac{n-2}{2s}} \left(\int |D\mathbf{v}|^2\right)^{\frac{1}{2}} \\
&\le c_n^{\frac{n}{2s}}\,||\mathbf{b}||_{L^s(\Omega)} \left(\int |\mathbf{u}|^2\right)^{\frac{s-n}{2s}} \left(\int |D\mathbf{u}|^2\right)^{\frac{n}{2s}} \left(\int |D\mathbf{v}|^2\right)^{\frac{1}{2}} \\
&\le c_n^{\frac{n}{2s}}\,||\mathbf{b}||_{L^s(\Omega)} \left[\left(1-\frac{n}{s}\right)\int |\mathbf{u}|^2 + \frac{n}{s}\int |D\mathbf{u}|^2\right]^{\frac{1}{2}} \left(\int |D\mathbf{v}|^2\right)^{\frac{1}{2}},
\end{aligned}$$

where we as usual interpret $\frac{1}{s}$ to be 0 in the case where $s = \infty$. Similarly,

$$\int |\mathbf{d}|\,|D\mathbf{u}|\,|\mathbf{v}| \le c_n^{\frac{n}{2t}}\,||\mathbf{d}||_{L^t(\Omega)} \left(\int |D\mathbf{u}|^2\right)^{\frac{1}{2}} \left[\left(1-\frac{n}{t}\right)\int |\mathbf{v}|^2 + \frac{n}{t}\int |D\mathbf{v}|^2\right]^{\frac{1}{2}},$$

and

$$\begin{aligned}
\int |\mathbf{V}|\,|\mathbf{u}|\,|\mathbf{v}| \le c_n^{\frac{n}{2p}}\,||\mathbf{V}||_{L^p(\Omega)} &\left[\left(1-\frac{n}{2p}\right)\int |\mathbf{u}|^2 + \frac{n}{2p}\int |D\mathbf{u}|^2\right]^{\frac{1}{2}} \\
&\left[\left(1-\frac{n}{2p}\right)\int |\mathbf{v}|^2 + \frac{n}{2p}\int |D\mathbf{v}|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Combining these inequalities, we see that

$$\mathcal{B}[\mathbf{u},\mathbf{v}] \le \left(\Lambda + c_n^{\frac{n}{2s}}\,||\mathbf{b}||_{L^s(\Omega)} + c_n^{\frac{n}{2t}}\,||\mathbf{d}||_{L^t(\Omega)} + c_n^{\frac{n}{2p}}\,||\mathbf{V}||_{L^p(\Omega)}\right)||\mathbf{u}||_{W^{1,2}(\Omega)^N}\,||\mathbf{v}||_{W^{1,2}(\Omega)^N}\,.$$

Therefore, we may take $\Gamma = \Lambda + c_n^{\frac{n}{2s}}\,||\mathbf{b}||_{L^s(\Omega)} + c_n^{\frac{n}{2t}}\,||\mathbf{d}||_{L^t(\Omega)} + c_n^{\frac{n}{2p}}\,||\mathbf{V}||_{L^p(\Omega)}$ so that (3.1.18), and therefore A5), holds. Clearly, the estimate from below on $\mathcal{B}[\mathbf{u},\mathbf{u}]$ may or may not be satisfied without further assumptions on the lower order terms. Thus, we have to assume that for some $\gamma > 0$, depending on $\lambda, \mathbf{V}, \mathbf{b}, \mathbf{d}$,

$$\gamma\left(||\mathbf{u}||^2_{L^2(\mathbb{R}^n)^N} + ||D\mathbf{u}||^2_{L^2(\mathbb{R}^n)^N}\right) \le \mathcal{B}[\mathbf{u},\mathbf{u}]\,.$$

In other words, we assume that (3.1.19) holds. This is valid, for instance, if $\mathbf{V}$ is positive definite and the first order terms are small with respect to the zeroth and second order terms. To be specific, we say that $\mathbf{V}$ is positive definite if there exists $\varepsilon > 0$ so that for any $\xi \in \mathbb{R}^N$, $V_{ij}(x)\,\xi^i\xi^j \ge \varepsilon\,|\xi|^2$ for every $x \in \Omega$. In this case,

$$\mathcal{B}[\mathbf{u},\mathbf{u}] \ge \lambda\int|D\mathbf{u}|^2 + \int \mathbf{b}^\alpha\,\mathbf{u}\cdot D_\alpha\mathbf{u} + \int \mathbf{d}^\beta D_\beta\mathbf{u}\cdot\mathbf{u} + \varepsilon\int|\mathbf{u}|^2\,.$$

If $\mathbf{b}$ and $\mathbf{d}$ are small in the sense that for some $\delta_1, \delta_2 > 0$

$$\left|\int \mathbf{b}^\alpha\,\mathbf{u}\cdot D_\alpha\mathbf{u} + \int \mathbf{d}^\beta D_\beta\mathbf{u}\cdot\mathbf{u}\right| \le \frac{\lambda}{1+\delta_1}\int|D\mathbf{u}|^2 + \frac{\varepsilon}{1+\delta_2}\int|\mathbf{u}|^2\,,$$

then it follows that $\mathcal{B}[\mathbf{u},\mathbf{u}] \ge \gamma\,||\mathbf{u}||^2_{W^{1,2}(\Omega)^N}$, where $\gamma = \min\left\{\frac{\lambda\delta_1}{1+\delta_1}, \frac{\varepsilon\delta_2}{1+\delta_2}\right\}$. There are other conditions that we could impose to ensure that the lower bounds holds for some $\gamma > 0$. When $N = 1$, the lower bound holds also in the presence of more involved non-degeneracy assumptions on the zeroth and first order terms that we discuss below.

By Lemma 3.3.1, the Caccioppoli inequality, (3.1.20), holds with C depending on n, s, t, γ, Λ, $||\mathbf{b}||_{L^s(\Omega)}$, and $||\mathbf{d}||_{L^t(\Omega)}$.

Moving towards (IB), (BB), and (H), when $N = 1$,

$$\mathcal{L}u = -D_\alpha\left(A^{\alpha\beta}D_\beta u + b^\alpha u\right) + d^\beta D_\beta u + Vu. \tag{3.6.2}$$

where $\lambda\,|\xi|^2 \le A^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta \le \Lambda\,|\xi|^2$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n$, $V \in L^p(\Omega)$, $b^\alpha \in L^s(\Omega)$, and $d^\beta \in L^t(\Omega)$. Moreover,

$$\mathcal{B}[u,v] = \int A^{\alpha\beta}D_\beta u\;D_\alpha v + b^\alpha\,u\;D_\alpha v + d^\beta D_\beta u\;v + V\,u\,v. \tag{3.6.3}$$

Since $u \in L^2(\Omega) \cap L^{2^*}(\Omega)$ and $Du \in L^2(\Omega)$, then by an application of the Hölder inequality $D|u|^2 = 2u\,Du \in L^p(\Omega)$ for any $p \in \left[1, \dfrac{n}{n-1}\right]$. It follows that $D_\alpha b^\alpha$ and $D_\beta d^\beta$ can be paired with $|u|^2$ in the sense of distributions. That is,

$$\begin{aligned}\mathcal{B}[u,u] &= \int A^{\alpha\beta} D_\beta u\, D_\alpha u + \frac{1}{2} b^\alpha\, D_\alpha |u|^2 + \frac{1}{2} d^\beta D_\beta |u|^2 + V\,|u|^2 \\ &= \int A^{\alpha\beta} D_\beta u\, D_\alpha u + \left(V - \frac{1}{2} D_\alpha b^\alpha - \frac{1}{2} D_\beta d^\beta\right)|u|^2, \end{aligned} \tag{3.6.4}$$

where the integrals above are interpreted as pairings in dual spaces.

Note that to ensure coercivity of the bilinear form, it suffices, for example, to assume that there exists $\delta > 0$ so that $V - \frac{1}{2} D_\alpha b^\alpha - \frac{1}{2} D_\beta d^\beta \geq \delta$ in the sense of distributions. That is, for any $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \geq 0$,

$$\int \left(V - \frac{1}{2} D_\alpha b^\alpha - \frac{1}{2} D_\beta d^\beta - \delta\right)\varphi \geq 0.$$

In this case, we see from (3.6.4) that the bound from below, (3.1.19), holds with $\gamma = \min\{\lambda, \delta\}$. If we further assume that $V - D_\alpha b^\alpha \geq 0$ and $V - D_\beta d^\beta \geq 0$ in Ω in the sense of distributions, then Lemma 3.4.1 implies that (IB) and (BB) hold for this setting with $C = C\left(n, q, s, t, \ell, \gamma, \Lambda, ||b||_{L^s(\Omega)}, ||d||_{L^t(\Omega)}\right)$ in (3.2.47) and (3.2.90) as long as B1) holds. If B2) also holds, then it follows from Lemma 3.5.6 that assumption (H) is also valid.

Therefore, we need to show that assumptions B1) and B2) are valid for $\mathbf{F}(\Omega) = W^{1,2}(\Omega)$. These facts are commonly used in the classical arguments for de Giorgi-Nash-Moser estimates, but the proofs are often omitted. One can find details, e.g., in [25]. Since Ω_R is bounded, then Lemma 2.3.27 implies that $W^{1,2}(\Omega)$ could also be defined as a completion of $C^\infty(\Omega_R)$ in the $W^{1,2}(\Omega)$-norm, thereby coinciding with the Sobolev space $H^{1,2}(\Omega; dx)$ of [25]. Then, given that $u \in W^{1,2}(\Omega_R)$, Theorem 1.20 of [25] implies that $(u-k)_+ \in W^{1,2}(\Omega)$, and therefore $(u-k)_+\zeta \in W^{1,2}(\Omega)$, $(u-k)_+ D_\alpha \zeta \in L^2(\Omega)$, $i = 1, ..., n$ (by a direct computation). Also, since we assume that $u\zeta \in W_0^{1,2}(\Omega)$, then $(u\zeta)_+ \in W_0^{1,2}(\Omega)$ by Lemma 1.23 of [25]. Finally, if ζ and k are non-negative, $0 \leq (u-k)_+\zeta \leq (u\zeta)_+$ and hence, $(u-k)_+\zeta \in W_0^{1,2}(\Omega)$ by Lemma 1.25(ii) of [25], as desired.

To show that B2) holds, we use a modification of the arguments given in Theorem 1.18 of [25]. We work with $f(t) = (t+k)^{-\omega}$, $t \geq 0$, which belongs to $C^1(\overline{\mathbb{R}_+})$ and has a bounded derivative on $\mathbb{R}^+$ (not on the entire $\mathbb{R}$). The exact same argument applies upon observing that a non-negative function $u \in W^{1,2}(B_R)$ can be approximated by non-negative $u_i \in C^\infty(B_R)$ due to Corollary 2.3.28.

3.6.3 Reverse Hölder potentials

Recall that B_p, $1 < p < \infty$, denotes the reverse Hölder class of all (real-valued) nonnegative locally L^p integrable functions that satisfy the reverse Hölder inequality. That is, $V \in B_p$ if $V \geq 0$ and there exists a constant C so that for any ball $B \subset \mathbb{R}^n$,

$$\left(\fint_B w(x)^p \, dx \right)^{1/p} \leq C \fint_B w(x) \, dx. \tag{3.6.5}$$

See Section 2.3.2 for more details.

For an $N \times N$ matrix function $\mathbf{M}(x)$, define lower and upper bounds on $\mathbf{M}$ in the following way

$$\begin{aligned} M_\ell(x) &= \inf \left\{ M_{ij}(x) \xi_j \xi_i : \xi \in \mathbb{R}^N, |\xi| = 1 \right\} \\ M_u(x) &= \sup \left\{ |M_{ij}(x) \xi_j \zeta_i| : \xi, \zeta \in \mathbb{R}^N, |\xi| = 1 = |\zeta| \right\}. \end{aligned}$$

For the zeroth order term $\mathbf{V}$, we assume that there exist constants $c_1, c_2 > 0$ and a non-trivial $V \in B_p$ for some $p \in \left[\frac{n}{2}, \infty\right)$ (and therefore $p \in \left(\frac{n}{2}, \infty\right)$ without loss of generality) so that

$$c_1 V \leq V_\ell \leq V_u \leq c_2 V. \tag{3.6.6}$$

Even if Ω is a proper subset of $\mathbb{R}^n$, we still assume that $\mathbf{V}$ is associated to some $V \in B_p$ which is defined on all of $\mathbb{R}^n$. As V is assumed to be non-trivial, it follows from the doubling property that V cannot vanish on any open set. We set $\mathbf{b}, \mathbf{d} = \mathbf{0}$.

One might wonder whether an appropriate matrix B_p class could be suitable in this context. We did not pursue this topic, in part, because the theory of matrix reverse Hölder classes seems to be largely undeveloped and developing the theory of matrix B_p for $p \neq 2$ was not in the scope of the present work. For the case of $p = 2$, some (very limited) discussion can be found in [42].

The space $\hat{W}^{1,2}_V(\Omega)$ serves as an alternative (but not equivalent) Hilbert space to $W^{1,2}_V(\Omega)$ for the case of reverse Hölder zeroth order terms. The spaces $\hat{W}^{1,2}_{0,V}(\Omega)$ and $W^{1,2}_{0,V}(\Omega)$ are the same – see Section 2.3.3. In practice, we find it easier to work with

$W_V^{1,2}(\Omega)$ compared to $\hat{W}_V^{1,2}(\Omega)$ due to the fact that $W_V^{1,2}(\Omega)$ coincides with the usual Sobolev spaces $W^{1,2}(\Omega)$ whenever Ω is bounded.

For $\mathbf{V}$ specified above, we set $\mathbf{F}(\Omega) = W_V^{1,2}(\Omega)^N$ and $\mathbf{F}_0(\Omega) = W_{0,V}^{1,2}(\Omega)^N$. Recall that the inner product on $W_V^{1,2}(\Omega)^N$ is given by

$$\langle \mathbf{u}, \mathbf{v} \rangle_{W_V^{1,2}(\Omega)^N} := \int_\Omega D_\alpha u^i D_\alpha v^i + u^i v^i m(\cdot, V)^2 .$$

As above, A1) and A2) follow directly from the definition. A3) is shown using the exact same argument as that for Lemma 2.3.1. For $\mathbf{u} \in W_V^{1,2}(\Omega)^N$ and $\xi \in C_c^\infty(U)$, it follows from the boundedness of ξ and $D\xi$, along with Remark 2.3.17, that $\mathbf{u}\xi \in W_V^{1,2}(\Omega \cap U)^N$ with $||\mathbf{u}\xi||_{W_V^{1,2}(\Omega\cap U)} \le C_\xi ||\mathbf{u}||_{W_V^{1,2}(\Omega)}$. Using the density of smooth functions in $W_V^{1,2}(U)^N$ for any bounded domain U, i.e., Lemma 2.3.27, the remainder of A4) follows from the arguments in Case 1 and Case 2, with appropriate modifications to (3.6.1).

The next goal is to show that boundedness and coercivity given by (3.1.18) and (3.1.19) hold. At this point we recall Lemma 2.3.25. Having that at hand, for any $\mathbf{u}, \mathbf{v} \in W_{0,V}^{1,2}(\Omega)$ we have

$$\begin{aligned}
\mathcal{B}[\mathbf{u}, \mathbf{v}] &= \int \mathbf{A}^{\alpha\beta} D_\beta \mathbf{u} \cdot D_\alpha \mathbf{v} + \mathbf{V}\,\mathbf{u} \cdot \mathbf{v} \le \Lambda \int |D\mathbf{u}|\,|D\mathbf{v}| + c_2 \int V\,|\mathbf{u}|\,|\mathbf{v}| \\
&\le \Lambda \left(\int |D\mathbf{u}|^2 \right)^{\frac{1}{2}} \left(\int |D\mathbf{v}|^2 \right)^{\frac{1}{2}} + c_2 \left(\int V\,|\mathbf{u}|^2 \right)^{\frac{1}{2}} \left(\int V\,|\mathbf{v}|^2 \right)^{\frac{1}{2}} \\
&\le (\Lambda + c_2\, C_{V,n})\, ||\mathbf{u}||_{W_V^{1,2}(\Omega)^N}\, ||\mathbf{v}||_{W_V^{1,2}(\Omega)^N} ,
\end{aligned}$$

where the last line follows from Lemma 2.3.25. Therefore, boundedness holds with $\Gamma = \Lambda + C\, C_{V,n}$. Since

$$\begin{aligned}
\mathcal{B}[\mathbf{u}, \mathbf{u}] &= \int \mathbf{A}^{\alpha\beta} D_\beta \mathbf{u} \cdot D_\alpha \mathbf{u} + \mathbf{V}\,\mathbf{u} \cdot \mathbf{u} \ge \lambda \int |D\mathbf{u}|^2 + c_1 \int V\,|\mathbf{u}|^2 \\
&\ge \frac{\lambda}{2} \int |D\mathbf{u}|^2 + \min\left\{ \frac{\lambda}{2}, c_1 \right\} \left[\int |D\mathbf{u}|^2 + \int V\,|\mathbf{u}|^2 \right],
\end{aligned}$$

then by another application of Lemma 2.3.25, we see that coercivity holds with $\gamma = \min\left\{ \frac{\lambda}{2}, \frac{\lambda}{2C}, \frac{c_1}{C} \right\}$.

By Lemma 3.3.1, (3.1.20) holds with $C = C(n, \gamma, \Lambda)$.

When $N = 1$, $\mathcal{L}$ and $\mathcal{B}$ are given by (3.6.2) and (3.6.3), respectively, with $b, d = 0$ and $V \in B_p$ for some $p \in \left(\frac{n}{2}, \infty\right)$, without loss of generality. By the non-negativity of V, Lemma 3.4.1 implies that (IB) and (BB) hold for this setting with $C = C(n, q, \ell, \gamma, \Lambda)$ in (3.2.47) and (3.2.90) whenever B1) holds. Since $V \in L^p_{loc}$, Lemma 3.5.6 shows that assumption (H) holds under the additional assumption of B2). In turn, B1) and B2) in the setting of $\mathbf{F}(\Omega) = W^{1,2}_V(\Omega)^N$ follow directly from the same statements for $\mathbf{F}(\Omega) = W^{1,2}(\Omega)^N$, i.e., Case 2, and Remark 2.3.17 since Ω_R and B_R are bounded and the statements B1) and B2) are qualitative (they assure membership in the corresponding function spaces, without particular norm control).

Remark 3.6.1. We point out that for the case of equations ($N = 1$) with the potentials in B_p class for some $p \in \left[\frac{n}{2}, \infty\right)$, a stronger version of the Harnack inequality than Lemma 3.5.5 is possible, without the dependence of constants on the size of the ball [12]. In the present paper, we do not need this stronger estimate, and we aim to keep the discussion uniform across several cases.

From the above arguments, we conclude that $\mathbf{\Gamma}(x, y)$ exists and satisfies the estimates of Theorem 3.2.6, where in the vector case ($N > 1$) we must assume that (IB) and (H) hold for solutions.

The estimates of Theorem 3.2.6 imply immediately that

$$\mathbf{\Gamma}(\cdot, y) \in Y^{1,2}\left(\mathbb{R}^n \setminus B_r(y)\right)^{N \times N} \quad \text{for any } r > 0.$$

With these estimates, however, it does not follow that $\mathbf{\Gamma}(\cdot, y) \in \mathbf{F}\left(\mathbb{R}^n \setminus B_r(y)\right)$ for the general space $\mathbf{F}$. Nevertheless, in many reasonable cases it is true. In Case 1, it follows clearly since $\mathbf{F}\left(\mathbb{R}^n \setminus B_r(y)\right) = Y^{1,2}\left(\mathbb{R}^n \setminus B_r(y)\right)^N$. In Case 2, it is true locally – i.e., we have that $\mathbf{\Gamma}(\cdot, y) \in W^{1,2}_{\text{loc}}\left(\mathbb{R}^n \setminus \{y\}\right)^{N \times N}$ because of the relationship between the spaces (see Lemma 2.3.7). Furthermore, for $|U| < \infty$, the space $Y^{1,2}(U)$ embeds continuously into $W^{1,2}(U)$, so we have

$$||\mathbf{\Gamma}(\cdot, y)||_{W^{1,2}(U \setminus B_r(y))} \le C_U \, ||\mathbf{\Gamma}(\cdot, y)||_{Y^{1,2}(U \setminus B_r(y))} \le C_U C r^{1-\frac{n}{2}}, \quad \forall r > 0, \qquad (3.6.7)$$

where C is the constant of Theorem 3.2.6. In Case 3, observe that for $|U| < \infty$,

$$W^{1,2}_V(U)^{N \times N} \hookrightarrow W^{1,2}(U)^{N \times N} \hookrightarrow Y^{1,2}(U)^{N \times N}.$$

(see Remark 2.3.20). Thus a similar estimate to (3.6.7) holds in Case 3.

By the same reasoning, similar conclusions hold for $\mathbf{\Gamma}(x, \cdot)$, $\mathbf{G_D}(\cdot, y)$, and $\mathbf{G_D}(x, \cdot)$.

Chapter 4

Construction of the Green and Neumann-Green functions by the method of layer potentials

4.0.4 Motivation and main results

In this chapter, we consider a particular case of the homogeneous operator with complex coefficients on the upper half space and prove existence of the Green and Neumann-Green functions and weak-type estimate analogous to (3.2.95). The key novelty here is that we construct the Green and Neumann-Green functions using layer potentials, and thus the result depends on invertibility of the layer potential operators rather than the set assumptions in the previous chapter. In particular, formally speaking, we do not require condition (BB) of Section 3.2.4 for our global estimate. More importantly, we treat Neumann-Green function as well (hereafter simply called the Neumann function) – a subject that we have not addressed so far in the non-homogeneous scenario. Finally, this approach gives as a byproduct certain new layer potential and Green operator estimates which are interesting on their own right. To our knowledge, these ideas have been pioneered in [39] for strongly elliptic systems with symmetric, constant coefficients on bounded Lipschitz domains when $n = 3$. Here we assume simply complex, bounded, and measurable coefficients.

We take L to be an operator

$$L = -\operatorname{div} A\nabla, \tag{4.0.1}$$

define on $\mathbb{R}^n_+ = \{(x,t) : x \in \mathbb{R}^n, t \in (0,\infty)\}$, where A is an elliptic matrix with complex-valued bounded measurable coefficients, i.e.,

$$\lambda|\xi|^2 \leq \Re e \, \langle A(x)\xi, \xi\rangle \equiv \Re e \sum_{i,j=1}^{n} A_{ij}(x)\xi_j\bar{\xi}_i \quad \text{and} \quad \|A\|_{L^\infty(\Omega)} \leq \Lambda, \tag{4.0.2}$$

for some $0 < \lambda, \Lambda < \infty$ and for all $\xi \in \mathbb{C}^n$, $x \in \mathbb{R}^n_+$, and solutions are interpreted in the standard weak sense. As we pointed out before, such an L can be viewed as a 2x2 elliptic system from the previous chapter, but our additional assumptions on regularity of solutions will be slightly different. At this point we assume, furthermore, that the coefficients of A are t-independent.

Let us discuss the assumption of t-independence of the coefficients. We are planning to use well-posedness of boundary value problems to pass from the estimates on the fundamental solution to the estimates on the Green and Neumann functions. It is known that some regularity in the transverse direction is required, e.g., in order to achieve solvability for the Dirichlet problem. For example, it is shown in [11] that for any function, ζ, satisfying

$$\int_0^1 (\zeta(\tau))^2 d\tau/\tau = +\infty,$$

there exists a real, symmetric, elliptic matrix, $A(x', t)$, whose modulus of continuity in t,

$$\omega(\tau) := \sup_{x' \in \mathbb{R}^{n-1},\, 0<t<\tau} |A(x', t) - A(x', 0)|,$$

is controlled by $\zeta(\tau)$, yet the corresponding elliptic-harmonic measure is completely singular with respect to Lebesgue measure. Thus, t-independence is a reasonable starting place for the study of such operators.

We will require the de Giorgi-Nash-Moser estimates in the spirit of the previous chapter. Specifically, we assume that (3.2.47) holds and the classical (stronger) form of the Hölder continuity assumption (3.2.48). Namely, we replace (3.2.48) with the following: there exist constants C and η, depending only on p, the dimension, and the ellipticity parameters of L, such that all weak solutions, u, of $Lu = 0$ on Ω, satisfy

$$\sup_{y,z\in B_{r/2},y\neq z} |u(y) - u(z)| \leq C \left(\frac{|y-z|}{r}\right)^{\eta} \left(\fint_{B_{2r}(x)} |u|^2\right)^{\frac{1}{2}}, \tag{4.0.3}$$

for all $y, z \in B_r(x)$, $x \in \Omega$, $r > 0$ such that $B_{2r}(x) \subset \Omega$. Note that (4.0.3) implies (3.2.47).

For convenience, throughout the chapter we will reference the "standard assumptions" on the differential operator L, whereby we mean

1. L is the homogeneous operator given by (4.0.1) with A an $n\times n$ matrix of complex-valued, L^∞, t-independent coefficients defined on $\mathbb{R}^n_+$ satisfying (4.0.2), and

2. the de Giorgi-Nash-Moser estimates (3.2.47) and (4.0.3) hold.

By Theorem 3.2.6, when L and L^* satisfy the standard assumptions, the fundamental solutions corresponding to L and L^* exist and satisfy (3.2.55)–(3.2.62).

With this at hand, one can tackle the mapping properties of the layer potentials. To chart a general plan, we start with formal definitions, the precise details will be presented below. The single and double layer potential operators are formally given by

$$\mathcal{S}_t f(x') := \int_{\mathbb{R}^{n-1}} \Gamma(x', t; y', 0) f(y') dy', \quad t \in \mathbb{R}, \quad \text{and} \tag{4.0.4}$$

$$\mathcal{D}_t f(x') := \int_{\mathbb{R}^{n-1}} \overline{\left(\partial_{\nu^*}\Gamma^*(\,\cdot\,,\,\cdot\,; x', t)\right)}(y', 0) f(y') dy', \quad t \neq 0. \tag{4.0.5}$$

Here, ∂_{ν^*} denotes the conormal derivative with respect to $\mathbb{R}^n_+$ associated to the adjoint matrix A^*, which is roughly

$$\left(\partial_{\nu^*}\Gamma^*(\,\cdot\,,\,\cdot\,; x', t)\right)(y', 0) = -e_n \cdot A^*(y')\nabla_{y',s}\Gamma^*(y', s, x', t)|_{s=0}.$$

The precise meaning is given by Lemma 4.0.33.

In order to study the layer potentials on the boundary, we define the operators

$$\begin{aligned} Kf(x') &:= \text{“}p.v.\text{”} \int_{\mathbb{R}^{n-1}} \overline{\left(\partial_{\nu^*}\Gamma^*(\,\cdot\,,\,\cdot\,, x', 0)\right)}(y', 0) f(y') dy', \quad x' \in \mathbb{R}^{n-1}, \\ \widetilde{K} f(x') &:= \text{“}p.v.\text{”} \int_{\mathbb{R}^{n-1}} \left(\partial_{\nu}\Gamma(x', 0, \,\cdot\,, \,\cdot\,)\right)(y', 0) f(y') dy', \quad x' \in \mathbb{R}^{n-1}, \end{aligned} \tag{4.0.6}$$

where "p.v." stands for the "principal value" of the integral, interpreted loosely for the moment. We use $\widetilde{K}$ instead of the traditional K^* to avoid confusion with the adjoint of K, which may not be equivalent if L is not self-adjoint. The trace of $\mathcal{D}_{\pm t}f$, $t > 0$, is given by

$$\mathcal{D}_0 f(x') = \left(-\frac{1}{2}I + K\right) f(x'), \tag{4.0.7}$$

for $x' \in \mathbb{R}^{n-1}$, and the normal derivative of the single layer potential,

$$\partial_\nu S^{\pm} f = \left(\mp\frac{1}{2}I + \widetilde{K}\right) f,$$

where $S_t^{\pm} f = S_{\pm t} f$, $t > 0$. In fact, the normal derivative for the solutions (in particular, for the single layer) is defined on its own right and the operator $\widetilde{K}$ is defined to satisfy the relationship above, once the jump formulas are established. We will give precise definitions in Section 4.1.1 and, in particular, prove in what sense $\mathcal{D}_0 f$ may be considered to be the trace of $\mathcal{D}_t f$. The tangential derivatives of the single layer are continuous across the boundary and we will denote the boundary trace of S_t by S_0. All these operators will be carefully defined below.

We are able to construct solutions to $Lu = 0$ via layer potentials. Indeed, we see (at least formally) by passing the operator under the integral in (4.0.4) that for a function g defined on the boundary $\mathbb{R}^{n-1}$,

$$u(x) = (\mathcal{S}_t g)(x') \tag{4.0.8}$$

solves the boundary problem

$$\begin{cases} Lu = 0 & \text{in } \mathbb{R}^n_+, \\ \text{Tr } u = \mathcal{S}_0 g & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{4.0.9}$$

Furthermore, if f is some prescribed boundary data on $\mathbb{R}^{n-1}$, and $\mathcal{S}_0$ is invertible in some appropriate sense, we may reformulate (4.0.8)-(4.0.9) with $g = \mathcal{S}_0^{-1} f$ to obtain the desired boundary values. In other words,

$$u(x) = \mathcal{S}_t[(\mathcal{S}_0)^{-1} f](x') \tag{4.0.10}$$

solves

$$\begin{cases} Lu = 0 & \text{in } \mathbb{R}^n_+, \\ \text{Tr } u = f & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{4.0.11}$$

Similarly, assuming $\partial_\nu \mathcal{S}$ is invertible in the appropriate sense,

$$u(x) = \mathcal{S}_t[(\partial_\nu \mathcal{S})^{-1} f](x') \tag{4.0.12}$$

solves

$$\begin{cases} Lu = 0 & \text{in } \mathbb{R}^n_+, \\ \partial_\nu u = f & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{4.0.13}$$

Motivated by formulas (4.0.10) and (4.0.12), one may construct the Green and Neumann functions using layer potentials. The Green function is given by

$$G_D(x; y) = \Gamma(x; y) - \phi_y(x), \tag{4.0.14}$$

where ϕ_y solves

$$\begin{cases} L\phi_y = 0 & \text{in } \mathbb{R}^n_+, \\ \mathrm{Tr}\ \phi_y = \Gamma(\,\cdot\,, 0; y) & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{4.0.15}$$

Similarly, the Neumann function is given by

$$G_N(x; y) = \Gamma(x; y) - \psi_y(x), \tag{4.0.16}$$

where ψ_y solves

$$\begin{cases} L\psi_y = 0 & \text{in } \mathbb{R}^n_+, \\ \partial_\nu \psi_y = \partial_{\nu_x} \Gamma(\,\cdot\,, 0; y) & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{4.0.17}$$

By formulas (4.0.11) and (4.0.13), we represent the solutions ϕ_y and ψ_y by

$$\phi_y(x) = \mathcal{S}_t[\mathcal{S}_0^{-1}(\Gamma(\,\cdot\,, 0; y))](x') \tag{4.0.18}$$

$$\psi_y(x) = \mathcal{S}_t[(\partial_\nu \mathcal{S})^{-1}(\partial_\nu \Gamma(\,\cdot\,, 0; y))](x'). \tag{4.0.19}$$

Combining (4.0.14)-(4.0.19), we have the following layer potential representations of the Green and Neumann functions:

$$G_D(x; y) = \Gamma(x; y) - \mathcal{S}_t[\mathcal{S}_0^{-1}(\Gamma(\,\cdot\,, 0; y))](x'), \tag{4.0.20}$$

$$G_N(x; y) = \Gamma(x; y) - \mathcal{S}_t[(\partial_\nu \mathcal{S})^{-1}(\partial_\nu \Gamma(\,\cdot\,, 0; y))](x'). \tag{4.0.21}$$

The resulting Green and Neumann functions rely on boundedness and invertibility of the involved operators, but they do not rely directly on, e.g., the regularity of solutions near the boundary, unlike the previous constructions. We now address the somewhat tricky issue of compatibility. It may be that the constructions in (4.0.20) and (4.0.21) produce Green and Neumann functions that differ from those constructed via Lax-Milgram and averaging as in [27] or in our Chapter 3 (for Dirichlet data), and [13], [33] (for Neumann data). Essentially, this can happen because the uniqueness granted by the Lax-Milgram Lemma is only uniqueness within the chosen Hilbert space.

The key to resolving the issue is to have compatible invertibility of $\mathcal{S}_0^{-1}$ and $(\partial_\nu \mathcal{S})^{-1}$. For given Banach function spaces $X(\mathbb{R}^{n-1})$ and $Y(\mathbb{R}^{n-1})$, we say that

$$\mathcal{S}_0 : X(\mathbb{R}^{n-1}) \to Y(\mathbb{R}^{n-1})$$

is *compatibly invertible* if, whenever $f \in Y(\mathbb{R}^{n-1}) \cap \dot{L}^2_{1/2}(\mathbb{R}^{n-1})$, $\mathcal{S}_0^{-1} f$ is the same in $X(\mathbb{R}^{n-1}) \cap \dot{L}^2_{-1/2}(\mathbb{R}^{n-1})$ whether we take the inverse of $\mathcal{S}_0 : X(\mathbb{R}^{n-1}) \to Y(\mathbb{R}^{n-1})$ or the inverse of $\mathcal{S}_0 : \dot{L}^2_{-1/2}(\mathbb{R}^{n-1}) \to \dot{L}^2_{1/2}(\mathbb{R}^{n-1})$. We say that

$$\partial_\nu \mathcal{S} : X(\mathbb{R}^{n-1}) \to Y(\mathbb{R}^{n-1})$$

is *compatibly invertible* if, whenever $f \in Y(\mathbb{R}^{n-1}) \cap L^2(\mathbb{R}^{n-1})$, $(\partial_\nu \mathcal{S})^{-1} f$ is the same whether we take the inverse of $\partial_\nu \mathcal{S} : X(\mathbb{R}^{n-1}) \to Y(\mathbb{R}^{n-1})$ or of $\partial_\nu \mathcal{S} : \dot{L}^2_{-1/2}(\mathbb{R}^{n-1}) \to L^2_{-1/2}(\mathbb{R}^{n-1})$. This guarantees that ϕ_y and ψ_y in (4.0.18)-(4.0.19) are the unique $\dot{W}^{1,2}(\mathbb{R}^n_+)$ solutions to (4.0.15) and (4.0.17), and therefore the resulting Green and Neumann functions agree with those constructed in [27] and [13] using Lax-Milgram. We note that

$$\mathcal{S}_0 : \dot{L}^2_{-1/2}(\mathbb{R}^{n-1}) \to \dot{L}^2_{1/2}(\mathbb{R}^{n-1})$$

and

$$\partial_\nu \mathcal{S} : \dot{L}^2_{-1/2}(\mathbb{R}^{n-1}) \to L^2_{-1/2}(\mathbb{R}^{n-1})$$

are invertible whenever L and L^* satisfy the standard assumptions [9, Theorems 2.32 and 3.16].

We now state our main results.

Theorem 4.0.22. *Let L, L^* satisfy the standard assumptions. If $\mathcal{S}_0 : H^{1,\infty}(\mathbb{R}^{n-1}) \to \dot{H}^{1,\infty}_1(\mathbb{R}^{n-1})$ is compatibly invertible, then*

$$\nabla_x G_D(\,\cdot\,; y) \in L^{\frac{n}{n-1},\infty}(\mathbb{R}^n_+) \text{ uniformly in } y \in \mathbb{R}^n_+. \tag{4.0.23}$$

If $-\frac{1}{2}I + \widetilde{K}$ is compatibly invertible on $H^{1,\infty}(\mathbb{R}^{n-1})$, then

$$\nabla_x G_N(\,\cdot\,; y) \in L^{\frac{n}{n-1},\infty}(\mathbb{R}^n_+) \text{ uniformly in } y \in \mathbb{R}^n_+. \tag{4.0.24}$$

This chapter employs much of the method of [39]. In that work, (4.0.23), (4.0.24), (4.0.28), and (4.0.29) were proven for strongly elliptic systems with symmetric, constant coefficients on bounded Lipschitz domains when $n = 3$. However, we assume simply complex, bounded, and measurable coefficients. This gives rise to an array of technicalities, such as justification of the proper Green representation formula, a (different) proof of the pointwise estimates on the conormal and tangential derivatives of the fundamental solution at the boundary, the issues of uniqueness of solutions and compatibility of inverses for layer potentials, and others. We note that treating $\mathbb{R}^n_+$ in place of a Lipschitz graph domain does not yield any loss of generality, since in our general context one can always reduce to this case (see the discussion in [2, p. 4539]).

We further aim to understand the mapping properties of operators

$$\mathbb{G}_D f(x) := \int_{\mathbb{R}^n_+} G_D(x; y) f(y) dy \tag{4.0.25}$$

and

$$\mathbb{G}_N f(x) := \int_{\mathbb{R}^n_+} G_N(x; y) f(y) dy, \tag{4.0.26}$$

which represent solutions to $Lu = f$ with the zero Dirichlet and Neumann boundary data, respectively.

Corollary 4.0.27. *Let L and L^* satisfy the standard assumptions. If $\mathcal{S}_0 : H^{1,\infty}(\mathbb{R}^{n-1}) \to \dot{H}^{1,\infty}_1(\mathbb{R}^{n-1})$ is compatibly invertible, then the solution operator*

$$\nabla \mathbb{G}_D : L^1(\mathbb{R}^n_+) \to L^{\frac{n}{n-1},\infty}(\mathbb{R}^n_+) \tag{4.0.28}$$

is bounded. If $-\frac{1}{2}I + \widetilde{K}$ is compatibly invertible on $H^{1,\infty}(\mathbb{R}^{n-1})$, then the solution operator

$$\nabla \mathbb{G}_N : L^1(\mathbb{R}^n_+) \to L^{\frac{n}{n-1},\infty}(\mathbb{R}^n_+) \tag{4.0.29}$$

is bounded.

Corollary 4.0.30. *Let L and L^* satisfy the standard assumptions. If $\mathcal{S}_0 : H^p(\mathbb{R}^{n-1}) \to \dot{H}^p_1(\mathbb{R}^{n-1})$ is compatibly invertible for $1 - \varepsilon < p < 1 + \varepsilon$ for some $\varepsilon > 0$, then (4.0.23) and (4.0.28) hold. If $-\frac{1}{2}I + \widetilde{K}$ is compatibly invertible on $H^p(\mathbb{R}^{n-1})$ for $1-\varepsilon < p < 1+\varepsilon$ for some $\varepsilon > 0$, then (4.0.24) and (4.0.29) hold.*

Proof. The corollary is reduced to Theorem 4.0.22 using interpolation of Hardy spaces. □

The conditions of Corollary 4.0.30 are known to be fulfilled in several cases. The operator $\mathcal{S}_0 : H^p(\mathbb{R}^{n-1}) \to \dot{H}^p_1(\mathbb{R}^{n-1})$ is compatibly invertible for $1-\varepsilon < p < 1+\varepsilon$ for some $\varepsilon > 0$ when the coefficients of the underlying elliptic operator are real, not necessarily symmetric by [26]. We remark that even in that case we do not know if $-\frac{1}{2}I + \widetilde{K}$ is compatibly invertible on $H^p(\mathbb{R}^{n-1})$ for $1-\varepsilon < p < 1+\varepsilon$ for some $\varepsilon > 0$. However, in the case of real coefficients, desired weak-type estimates are already known by [23] and [13], respectively.

Furthermore, in the realm of complex coefficient operators, we know that compatible well-posedness of the Neumann boundary value problem in L^p for some $p > 1$ extrapolates to compatible well-posedness of the Neumann boundary value problem in H^q for $1-\varepsilon < q < p$, for some $\varepsilon > 0$, at least in the presence of interior Hölder continuity of Dirichlet solutions for certain operators associated to L and L^* ([5]). In a sense, the latter condition should allow interior Hölder continuity for operators in $\mathbb{R}^n$ obtained by reflection across the boundary. In addition, compatible well-posedness yields desired invertibility of layer potentials by [9, Theorem 3.18]. And hence, at this stage, Corollary 4.0.30 could be applied to obtain weak-type estimates on the Green function.

It is interesting to point out that contrary to the case of real symmetric coefficients, we only expect well-posedness of boundary problems in H^p for p close to 1 (in the real symmetric case one has well-posedness in H^p for $1-\varepsilon < p < 2+\varepsilon$; however, in real non-symmetric case for every $p > 1$ there are counterexamples). Remarkably, it is exactly the results for p close to 1 which are exploited in our argument.

4.0.5 Conormal derivative

We discuss now the precise definition of the conormal derivative in the context of a single homogeneous equation on the upper half-space, $\mathbb{R}^n_+$, with complex bounded measurable coefficients. In this chapter, we only apply the conormal derivative in this context, so we have not included a generalization to non-homogeneous systems or other domains.

For $u \in C^1(\overline{\mathbb{R}^n_+})$, the conormal derivative of u with respect to $\mathbb{R}^n_+$ is defined

$$\partial_\nu u(x') := A(x')\nabla u(x',t)|_{t=0} \cdot \nu(x') = -(A\nabla u)_n(x',t)|_{t=0} \quad x' \in \mathbb{R}^{n-1}. \tag{4.0.31}$$

However, for general $u \in \dot{W}^{1,2}(\mathbb{R}^n_+)$, we cannot meaningfully discuss $A\nabla u$ on $\mathbb{R}^{n-1}$. Motivated by the integration by parts formula

$$\int_{\mathbb{R}^n_+} -\operatorname{div} A\nabla u \cdot \overline{\Psi} = \int_{\mathbb{R}^n_+} A\nabla u \cdot \overline{\nabla \Psi} - \int_{\mathbb{R}^{n-1}} (A\nabla u)_n \cdot \overline{\Psi}|_{t=0},$$

we restrict the definition to u in $\dot{W}^{1,2}$ in a neighborhood of $\mathbb{R}^{n-1}$ solving $Lu = 0$ in a neighborhood of $\mathbb{R}^{n-1}$, and we consider $\partial_\nu u$ to be a distribution defined by

$$\langle \partial_\nu u, \psi \rangle = \int_{\mathbb{R}^n_+} A\nabla u \cdot \overline{\nabla \Psi}, \tag{4.0.32}$$

for all $\psi \in C_c^\infty(\mathbb{R}^{n-1})$ and for $\Psi \in C_c^\infty(\overline{\mathbb{R}^n_+})$ any extension of ψ such that u is a solution on the support of Ψ. The next theorem defines $\partial_\nu u$ in full specificity.

Lemma 4.0.33. *Suppose $u \in \dot{W}^{1,2}(\mathbb{R}^n_\pm)$ solves $Lu = 0$ in a smooth domain $\Omega \subset \mathbb{R}^n_\pm$ with $\mathbb{R}^{n-1} \subset \partial\Omega$. Then there exists $\partial_\nu^\pm u \in \dot{L}^2_{-1/2}(\mathbb{R}^{n-1})$ such that*

$$\langle \partial_\nu^\pm u, \psi \rangle = \int_\Omega A\nabla u \cdot \overline{\nabla \Psi} \tag{4.0.34}$$

for all $\psi \in \dot{L}^2_{1/2}(\mathbb{R}^{n-1})$ and for any $\Psi \in \dot{W}^{1,2}(\mathbb{R}^n_\pm)$ such that

$$Tr\ \Psi = \begin{cases} \psi \text{ on } \mathbb{R}^{n-1}, \\ 0 \text{ on } \partial\Omega \setminus \mathbb{R}^{n-1}. \end{cases} \tag{4.0.35}$$

If $L^* u = 0$ in Ω, then there exists $\partial_{\nu^*}^\pm u$ in $\dot{L}^2_{-1/2}(\mathbb{R}^{n-1})$ defined by (4.0.34) with A^* replacing A.

Proof. First, we note that the existence of $\Psi \in \dot{W}^{1,2}(\mathbb{R}^n_\pm)$ satisfying (4.0.35) is well-known by standard trace/extension theory.

By the Riesz Representation Theorem, it suffices to show that (4.0.34) defines a bounded linear functional on $\dot{L}^2_{1/2}(\mathbb{R}^{n-1})$. Linearity is clear from the definition.

We must show that $\langle \partial_\nu^\pm u, \psi \rangle$ is well-defined. To this end, suppose $\Psi_1, \Psi_2 \in \dot{W}^{1,2}(\Omega \cap \mathbb{R}^n_\pm)$ and both satisfy (4.0.35). Then $\Psi_1 - \Psi_2 \in \dot{W}^{1,2}_0(\Omega)$. By the definition of a weak solution,

$$\int_\Omega A\nabla u \cdot \overline{\nabla(\Psi_1 - \Psi_2)} = 0.$$

It then follows by linearity that the functional is well-defined.

It remains to show boundedness. Since we have shown that we may choose any extension satisfying (4.0.35), we choose Ψ such that

$$||\Psi||_{\dot{W}^{1,2}(\Omega)} \leq C||\mathrm{Tr}\ \Psi||_{\dot{L}^2_{1/2}(\partial\Omega)} = C||\psi||_{\dot{L}^2_{1/2}(\mathbb{R}^{n-1})}.$$

We may do this since the trace operator has a bounded right-inverse. Then,

$$|\langle \partial_\nu u, \psi\rangle| \leq ||A||_{L^\infty}||u||_{\dot{W}^{1,2}}||\Psi||_{\dot{W}^{1,2}} \leq C||\psi||_{\dot{L}^2_{1/2}(\mathbb{R}^{n-1})}.$$

$\square$

We will use $\partial_\nu u$ to define the conormal derivative with respect to $\mathbb{R}^n_+$, i.e.,

$$\partial_\nu u := \partial_\nu^+ u. \tag{4.0.36}$$

For any u satisfying the hypotheses of Theorem 4.0.33 in both upper and lower half-spaces, we have

$$\partial_\nu^+ u = -\,\partial_\nu^- u. \tag{4.0.37}$$

To see this, note that for $\psi \in \dot{L}^2_{1/2}(\mathbb{R}^{n-1})$ we may choose an evenly reflected extension, $\Psi \in \dot{W}^{1,2}(\mathbb{R}^n)$, such that $Lu = 0$ on the support of Ψ. For details on this reflection, see Appendix A. Then,

$$\langle \partial_\nu^+ u, \psi\rangle + \langle \partial_\nu^- u, \psi\rangle = \int A\nabla u \cdot \overline{\nabla\Psi} = 0.$$

4.1 Layer potentials

4.1.1 Definitions and boundary traces

Recall the formal definition of layer potentials in (4.0.4) and (4.0.5). Rigorously, it can be proved that for $f \in L^2(\mathbb{R}^{n-1})$, we have $\mathcal{S}_t : \dot{L}^2_{-1/2}(\mathbb{R}^{n-1}) \to \dot{W}^{1,2}(\mathbb{R}^n)$ and $L(\mathcal{S}_t f) = 0$ for all $t > 0$. The former is, for example, a very particular instance of the result [9, Theorem 3.1]. Therefore, the trace of $S_t f$ exists in $\dot{L}^2_{1/2}(\mathbb{R}^{n-1})$ and the conormal derivatives $\partial_\nu^+ \mathcal{S} f$ and $\partial_\nu^- \mathcal{S} f$ exist in the sense of Lemma 4.0.33. Futhermore, as in [28, Lemma 3.2], we have for $f \in L^2(\mathbb{R}^{n-1})$,

$$\partial_\nu^+ \mathcal{S} f - \partial_\nu^- \mathcal{S} f = f.$$

We can now define operators K and $\widetilde{K}$, previously understood loosely by (4.0.6), as

$$\widetilde{K} := \frac{1}{2}I + \partial_\nu^+ \mathcal{S} \quad \text{and} \tag{4.1.1}$$

$$K := adj(\widetilde{K}_{L^*}), \tag{4.1.2}$$

where adj is the adjoint on $\mathbb{R}^{n-1}$ and $\widetilde{K}_{L^*}$ is $\widetilde{K}$ with respect to the operator L^*. For now, we define these operators only on L^2, but they extend to $f \in \dot{H}_1^p$ as seen in the next section. Similar considerations apply to the weak definition of the double layer potential. We refer the reader to [9] for the details and pass to the mapping properties that are of importance in the present work.

4.1.2 Boundedness of layer potentials

The next theorem justifies the definitions in the previous section for $f \in H^p(\mathbb{R}^{n-1})$, $\frac{n}{n+\alpha} < p < 2+\epsilon$. It will also be used to get certain operator bounds that will be needed to prove the main theorem. (4.1.7)-(4.1.11) were established in [28, Theorem 1.1], and (4.1.12) in [9, (7.4)].

Hereafter, "n.t.a.e." means that for a.e. $x \in \partial\Omega$ we have convergence along every path in the cone

$$\gamma(x) := \{y \in \Omega : |y-x| < \operatorname{dist}(y, \partial\Omega)\}, \tag{4.1.3}$$

$\mathcal{N}$ denotes the nontangential maximal operator

$$\mathcal{N}u(x) := \sup_{y\in\gamma(x)} |u(y)|, \tag{4.1.4}$$

and $\widetilde{\mathcal{N}}$ is the L^2-averaged nontangential maximal operator

$$\widetilde{\mathcal{N}}u(x) := \sup_{y\in\gamma(x)} \left(\fint_{|z-y|<\frac{1}{4}\operatorname{dist}(y,\partial\Omega)} |u(z)|^2 dz \right)^{1/2}. \tag{4.1.5}$$

Theorem 4.1.6. *Let L and L^* satisfy the standard assumptions. Then there exist $\alpha > 0$ and $p^+ > 2$ depending on the constants in the standard assumptions only such* *that for* $p_\alpha = \frac{n}{n+\alpha}$ *we have*

$$||\widetilde{\mathcal{N}}(\nabla \mathcal{S}_t f)||_{L^p(\mathbb{R}^n)} \leq C||f||_{H^p(\mathbb{R}^n)}, \tag{4.1.7}$$

$$\sup_{t>0} ||\nabla \mathcal{S}_t f||_{L^p(\mathbb{R}^n)} \leq C||f||_{H^p(\mathbb{R}^n)}, \tag{4.1.8}$$

$$||\nabla_{||} \mathcal{S}_0 f||_{H^p(\mathbb{R}^n)} \leq C||f||_{H^p(\mathbb{R}^n)}, \tag{4.1.9}$$

$$||\widetilde{K} f||_{H^p(\mathbb{R}^n)} \leq C||f||_{H^p(\mathbb{R}^n)}, \tag{4.1.10}$$

$$||\mathcal{N}(\mathcal{D}_t f)||_{L^q(\mathbb{R}^{n-1})} \leq C||f||_{L^q(\mathbb{R}^{n-1})}, \tag{4.1.11}$$

$$||\widetilde{\mathcal{N}}(\nabla \mathcal{D}_t f)||_{L^p(\mathbb{R}^{n-1})} \leq C||f||_{\dot{H}^p_1(\mathbb{R}^{n-1})}, \tag{4.1.12}$$

for all $p \in (p_\alpha, p^+)$ *and* $q \in (\frac{p^+}{p^+-1}, \infty)$*. Here, for every* $f \in H^p(\mathbb{R}^{n-1})$*,* $S_t f$ *converge to* $S_0 f$ *n.t.a.e. and* $\nabla_{||} S_t f$ *converge to* $\nabla_{||} S_0 f$ *in the sense of distributions. The convergence to* $\widetilde{K}$ *is encoded in the definition* (4.1.1).

This leads immediately to the following operator bounds:

Theorem 4.1.13. *Let* L *and* L^* *satisfy the standard assumptions. Then there exist* $\alpha > 0$ *and* $\epsilon > 0$ *depending on the constants in the standard assumptions only for* $\frac{n}{n+\alpha} < p < 2 + \epsilon$*, the following operators are bounded:*

$$\mathcal{S}_0 : H^{p,\infty}(\mathbb{R}^n) \to \dot{H}^{p,\infty}_1(\mathbb{R}^n), \tag{4.1.14}$$

$$\widetilde{K} : H^{p,\infty}(\mathbb{R}^n) \to H^{p,\infty}(\mathbb{R}^n), \tag{4.1.15}$$

$$\partial_\nu \mathcal{D} : \dot{H}^{p,\infty}_1(\mathbb{R}^n) \to H^{p,\infty}(\mathbb{R}^n). \tag{4.1.16}$$

Proof. (4.1.14) and (4.1.15) follow immediately by interpolation of (4.1.9) and (4.1.10).

Since $\mathcal{D}_t f$ is a solution of L, [28, Lemma 6.1] combined with (4.1.12) gives us that $\partial_\nu \mathcal{D} f$ exists, belongs to $H^p(\mathbb{R}^n)$, and

$$||\partial_\nu \mathcal{D} f||_{H^p(\mathbb{R}^{n-1})} \leq C||f||_{\dot{H}^p_1(\mathbb{R}^{n-1})}.$$

Interpolation now gives us (4.1.16). □

4.1.3 Trace of the double layer potential

Theorem 4.1.17. *Under the conditions of Theorem 4.1.13,* K*, as defined by (4.1.1) and (4.1.2), extends continuously to a bounded linear map*

$$K : \dot{H}^p_1(\mathbb{R}^{n-1}) \to \dot{H}^p_1(\mathbb{R}^{n-1}), \tag{4.1.18}$$

and for $f \in \dot{H}^p_1(\mathbb{R}^{n-1})$,

$$\mathcal{D}_t f \to (-\frac{1}{2}I + K)f \quad \textit{n.t.a.e.} \tag{4.1.19}$$

and tangential derivatives converge in the sense of distributions.

Proof. In [28, Lemma 3.4], we have that for $f \in L^2$,

$$\mathcal{D}_t f \to (-\frac{1}{2}I + K)f \quad \text{weakly in } L^2(\mathbb{R}^{n-1}). \tag{4.1.20}$$

Furthermore, combining [9, (7.4)] with [28, Lemmas 6.1 and 6.2], it follows that for $f \in \dot{H}^p_1(\mathbb{R}^{n-1})$, $1 < p < 2 + \epsilon$, there exists $g \in \dot{H}^p_1(\mathbb{R}^{n-1})$ such that

$$\begin{gathered} \mathcal{D}_t f \to g \quad \text{n.t.a.e.} \\ \nabla_{\|} \mathcal{D}_t f \to \nabla_{\|} g \quad \text{in the sense of distributions} \\ ||g||_{\dot{H}^p_1(\mathbb{R}^{n-1})} \leq C||\widetilde{\mathcal{N}}(\nabla \mathcal{D} f)||_{L^p(\mathbb{R}^{n-1})} \leq C||f||_{\dot{H}^p_1(\mathbb{R}^{n-1})}. \end{gathered} \tag{4.1.21}$$

Thus, to get from (4.1.21) to (4.1.19), (4.1.18) we must show that K extends to $\dot{H}^p_1(\mathbb{R}^{n-1})$ and that, for $f \in \dot{H}^p_1(\mathbb{R}^{n-1})$,

$$g = (-\frac{1}{2}I + K)f \quad \text{as equivalence classes in } \dot{H}^p_1(\mathbb{R}^{n-1}), \tag{4.1.22}$$

with g the trace from (4.1.21).

It would suffice to show (4.1.22) for $f \in C^\infty_c(\mathbb{R}^{n-1})$ as $C^\infty_c(\mathbb{R}^{n-1})$ is dense in $\dot{H}^p_1(\mathbb{R}^{n-1})$.

Take $f \in C^\infty_c(\mathbb{R}^{n-1})$. Then, from (4.1.20) and (4.1.21),

$$\mathcal{D}_t f \to (-\frac{1}{2}I + K)f \quad \text{weakly in } L^2(\mathbb{R}^{n-1}), \tag{4.1.23}$$

$$\mathcal{D}_t f \to g \quad \text{n.t.a.e.}, \tag{4.1.24}$$

and tangential derivatives converge in the sense of distributions.

Take $\vec{\phi} \in C^\infty_c(\mathbb{R}^{n-1})^{n-1}$. From (4.1.23) we have

$$\int_{\mathbb{R}^{n-1}} (\nabla_{\|} \mathcal{D}_t f) \cdot \vec{\phi} = -\int_{\mathbb{R}^{n-1}} (\mathcal{D}_t f) \cdot \operatorname{div}_{\|} \vec{\phi} \to -\int_{\mathbb{R}^{n-1}} (-\frac{1}{2}I + K)f \cdot \operatorname{div}_{\|} \vec{\phi}. \tag{4.1.25}$$

On the other hand, (4.1.24) gives

$$\int_{\mathbb{R}^{n-1}} (\nabla_{\|} \mathcal{D}_t f) \cdot \vec{\phi} \to \int_{\mathbb{R}^{n-1}} \nabla_{\|} g \cdot \vec{\phi} \tag{4.1.26}$$

Thus, for all $\vec{\phi} \in C_c^\infty(\mathbb{R}^{n-1})^{n-1}$,

$$\int_{\mathbb{R}^{n-1}} \nabla_{\|} g \cdot \vec{\phi} = \int_{\mathbb{R}^{n-1}} (-\frac{1}{2} I + K) f \cdot \operatorname{div}_{\|} \vec{\phi}. \tag{4.1.27}$$

Therefore, $\nabla_{\|} g$ is the distributional derivative of $(-\frac{1}{2} I + K) f$, and (4.1.22) follows. □

Therefore, we define $\mathcal{D}_0 f$ to be the trace (in the non-tangential a.e. sense) of $\mathcal{D}_t f$, and we have

$$\mathcal{D}_0 f = \Big(-\frac{1}{2} I + K \Big) f \quad \text{and} \quad ||\mathcal{D}_0 f||_{\dot{H}_1^p(\mathbb{R}^{n-1})} \leq C ||f||_{\dot{H}_1^p(\mathbb{R}^{n-1})} \tag{4.1.28}$$

for $f \in \dot{H}_1^p(\mathbb{R}^{n-1})$.

Lemma 4.1.29. *Under the conditions of Theorem 4.1.13,*

$$K : \dot{H}_1^{p,\infty}(\mathbb{R}^{n-1}) \to \dot{H}_1^{p,\infty}(\mathbb{R}^{n-1}) \tag{4.1.30}$$

is bounded.

Proof. This follows immediately from Theorem 4.1.17 and interpolation. □

4.1.4 $H^{1,\infty}$ bounds on $\nabla\Gamma$

To proceed, we will need a theorem from [2]. In it, $L := -\operatorname{div} A \nabla$ is a second order divergence form complex elliptic operator with bounded measurable coefficients defined on $\mathbb{R}^n$. Note that we do not need to assume the de Giorgi-Nash-Moser bound for solutions of L.

Theorem 4.1.31 ([2], Proposition 2.1)**.** *Suppose the matrix A is $t-$independent, i.e., A=A(x'). Then there is a uniform constant $\epsilon > 0$ depending only on n and ellipticity, and for every $p \in [2, 2+\epsilon)$, a uniform constant C_p such that, for each fixed cube $Q \subset \mathbb{R}^{n-1}$, and $t \in \mathbb{R}$, if $Lu = 0$ in the box $I_Q := 4Q \times (t - l(Q), t + l(Q))$, then we have the following estimates*

$$\left(\frac{1}{|Q|} \int_Q |\nabla u(x', t)|^p \, dx' \right)^{1/p} \leq C_p \left(\frac{1}{|Q^*|} \iint_{Q^*} |\nabla u(x', \tau)|^2 \, dx' \, d\tau \right)^{1/2}, \tag{4.1.32}$$

$$\left(\frac{1}{|Q|} \int_Q |\nabla u(x', t)|^p \, dx' \right)^{1/p} \leq C_p \left(\frac{1}{l(Q)^2} \frac{1}{|Q^{**}|} \iint_{Q^{**}} |u(x', \tau)|^2 \, dx' \, d\tau \right)^{1/2}, \tag{4.1.33}$$

where $Q^* := 2Q \times (t - l(Q)/4, t + l(Q)/4)$ is an n-dimensional rectangle with diameter comparable to that of Q, and $Q^{**} := 3Q \times (t - l(Q)/2, t + l(Q)/2)$ is a fattened version of Q^*.

With Theorem 4.1.31 in hand, we are ready to proceed. For $y = (y', s)$ let y^* be its reflection across $\mathbb{R}^{n-1}$,

$$y^* := (y', -s).$$

Define $\widetilde{\Gamma}_y(x)$ by the difference

$$\widetilde{\Gamma}_y(x) := \Gamma(x; y) - \Gamma(x; y^*). \tag{4.1.34}$$

Theorem 4.1.35. *Let L and L^* satisfy the standard assumptions. Then $\partial_\nu^{\pm}\widetilde{\Gamma}_y(\,\cdot\,, t)|_{t=0}$ and $\nabla_{||}\widetilde{\Gamma}_y(\,\cdot\,, t)|_{t=0}$ are in $L^1(\mathbb{R}^{n-1})$ uniformly in $y \in \mathbb{R}^n_+$.*

Proof. Let us prove first that $\nabla\widetilde{\Gamma}_y \in L^1(\mathbb{R}^{n-1})$ uniformly in $y \in \mathbb{R}^n_+$.

Let Q be the $(n-1)$-dimensional cube in $\mathbb{R}^{n-1}$ centered at $(y', 0)$ with $l(Q) = s$. Let $S_3 = 8Q$ and $S_k := 2^{k+1}Q \setminus 2^kQ$ for $k = 4, 5, 6, \ldots$. Then, by Holder's inequality,

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} |\nabla\widetilde{\Gamma}_y(x', 0)|\, dx' &= \sum_{k=3}^{\infty} \int_{S_k} |\nabla\widetilde{\Gamma}_y(x', 0)|\, dx' \\ &\leq C \sum_{k=3}^{\infty} (2^k s)^{(n-1)/2} \Big(\int_{S_k} |\nabla\widetilde{\Gamma}_y(x', 0)|^2\, dx' \Big)^{1/2}. \end{aligned} \tag{4.1.36}$$

By Theorem 4.1.31, we have the following estimate for each cube $P \subset S_k$:

$$\Big(\int_P |\nabla\widetilde{\Gamma}_y(x', 0)|^2\, dx' \Big)^{1/2} \leq C \Big(l(P)^{n-3} \frac{1}{|P^{**}|} \iint_{P^{**}} |\widetilde{\Gamma}_y(x', \tau)|^2\, dx'\, d\tau \Big)^{1/2},$$

where $P^{**} := 3P \times (t - l(P)/2, t + l(P)/2))$.

For $k = 3$, divide S_3 into essentially disjoint cubes $T_1, \ldots, T_M$ of side length s, where

M depends on dimension only. Then, using (3.2.60) and (4.1.33),

$$\begin{aligned}
\int_{S_3} |\nabla\widetilde{\Gamma}_y(x',0)|^2\,dx' &= \sum_{i=1}^{M}\int_{T_i} |\nabla\widetilde{\Gamma}_y(x',0)|^2\,dx' \\
&\le \sum_{i=1}^{M} Cs^{n-3}\frac{1}{|T_i^{**}|}\iint_{T_i^{**}} |\Gamma(x',\tau;y)-\Gamma(x',\tau;y^*)|^2\,dx'd\tau \\
&\le \sum_{i=1}^{M} Cs^{n-3}\frac{1}{|T_i^{**}|}\iint_{T_i^{**}} |(x',|\tau|)-(y',s)|^{-2n+4}\,dx\,d\tau \\
&\le \sum_{i=1}^{M} Cs^{n-3}\left(\frac{s}{2}\right)^{-2n+4} \\
&= Cs^{-n+1}.
\end{aligned} \tag{4.1.37}$$

For $k \geq 4$, S_k can be covered by essentially disjoint cubes $P_1, ..., P_m$, with m depending on dimension and with $l(P_i) = 2^{k-2}s$. Then,

$$P_i^{**} \subset 2^{k+2}Q \setminus 2^{k-2}Q \times (-2^k s, 2^k s) =: S_k^{**},$$

and

$$\begin{aligned}
\int_{S_k} |\nabla\widetilde{\Gamma}_y(x',0)|^2\,dx' &= \sum_{i=1}^{m}\int_{P_i} |\nabla\widetilde{\Gamma}_y(x',0)|^2\,dx \\
&\le \sum_{i=1}^{m} C(2^{k-2}s)^{n-3} \fint_{P_i^{**}} |\widetilde{\Gamma}_y(x',\tau)|^2\,dx'\,d\tau \qquad \text{by (4.1.33)} \\
&\le \sum_{i=1}^{m} C(2^{k-2}s)^{n-3} \sup_{x\in P_i^{**}} |\widetilde{\Gamma}_y(x)|^2 \\
&\le C(2^k s)^{n-3} \sup_{x\in S_k^{**}} |\widetilde{\Gamma}_y(x)|^2.
\end{aligned} \tag{4.1.38}$$

We wish to leverage the Hölder continuity estimate (4.0.3) on each cube S_k^{**}. $\Gamma(x;\cdot)$ is a solution to $L_y^*\Gamma(x;y) = 0$ for $x \neq y$. So for $k \geq 4$, choose $R_k = 2^{k-4}s$, and then $y, y^* \in B := B(y,0;R_k)$ and for all $x \in S_k^{**}$ and $z \in 2B$, we have $|x-z| \geq 2^{k-3}s$.

Therefore, (4.0.3) applied to $\Gamma(x;\,\cdot\,)$, $x \in S_k^{**}$ gives

$$\begin{aligned}
|\Gamma(x;y)-\Gamma(x;y^*)|^2 &\le C\left(\frac{|y-y^*|}{2^{k-4}s}\right)^{2\alpha} \fint_{2B} |\Gamma(x;z)|^2 dz \\
&\le C\left(\frac{2s}{2^{k-4}s}\right)^{2\alpha} \sup_{z\in 2B} |\Gamma(x;z)|^2 \\
&\le C2^{(5-k)2\alpha} \sup_{z\in 2B} |x-z|^{-2n+4} \\
&\le C2^{-2k\alpha}(2^{k-3}s)^{-2n+4} \\
&= C2^{-2k\alpha+k(-2n+4)}s^{-2n+4}.
\end{aligned} \tag{4.1.39}$$

Combining (4.1.38) and (4.1.39), we have for $k \ge 4$,

$$\int_{S_k} |\nabla\widetilde{\Gamma}_y(x',0)|^2\,dx' \le C(2^k s)^{n-3}2^{-2k\alpha+k(-2n+4)}s^{-2n+4} = Cs^{-n+1}2^{-k(n-1)-2k\alpha}. \tag{4.1.40}$$

Note that C can be chosen independent of k.

Finally, combining (4.1.36), (4.1.37), and (4.1.40), we get

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} |\nabla\widetilde{\Gamma}_y(x',0)|\,dx' \\
&\quad\le C\left[(8s)^{(n-1)/2}s^{-(n-1)/2} + \sum_{k=4}^{\infty}(2^k s)^{(n-1)/2}\left(s^{-(n-1)}2^{-k(n-1)-2k\alpha}\right)^{1/2}\right] \\
&\quad\le C\left[1+\sum_{k=4}^{\infty}2^{-k\alpha}\right],
\end{aligned}$$

where C depends only on dimension, ellipticity, and the constant in (4.0.3).

So far we have shown that $\nabla\widetilde{\Gamma}_y(\,\cdot\,,t)|_{t=0}$ is in $L^1(\mathbb{R}^{n-1})$ uniformly in $y \in \mathbb{R}^n_+$. It remains to derive the desired estimate on $\partial_\nu^{\pm}\widetilde{\Gamma}_y(\,\cdot\,,t)|_{t=0}$. To this end, recall the weak definition of the conormal derivative (cf. Lemma 4.0.33) and notice that Theorem 4.1.31 yields (under the same assumptions) estimates (4.1.32)–(4.1.33) with $\nabla u(x',t)$ on the left-hand side replaced by $\partial_\nu u(x',\tau)|_{\tau=t}$ on the left-hand side at least for $p=2$. This is because u is a solution and one can choose Ψ in the definition of the normal derivative to be supported in $Q\times(t-l(Q)/10, t+l(Q)/10)$, at which point (4.0.34) combines with (4.1.32)–(4.1.33) to yield the desired result. The remainder of the argument is the same. □

Theorem 4.1.41 (Green formula). *Suppose $u \in \dot{W}^{1,2}(\mathbb{R}^n_\pm)$ solves $Lu = f$ in $\mathbb{R}^n_\pm$ with $f \in C_c^\infty(\mathbb{R}^n_\pm)$. Then,*

$$u(x) = \int_{\mathbb{R}^n_\pm} f(y)\Gamma(x;y)dy + \mathcal{S}_t(\partial_\nu^\pm u)(x') - \mathcal{D}_t(\mathit{Tr}\ u)(x') \tag{4.1.42}$$

for a.e. $x = (x', t) \in \mathbb{R}^n_\pm$.

We note that $\mathcal{S}_t : \dot{L}^2_{-1/2}(\mathbb{R}^{n-1}) \to \dot{W}^{1,2}(\mathbb{R}^n)$ and $\mathcal{D}_t : \dot{L}^2_{1/2}(\mathbb{R}^{n-1}) \to \dot{W}^{1,2}(\mathbb{R}^n)$ are bounded so the operators in (4.1.42) are well-defined for the data at hand (cf., e.g., [9, Theorem 3.1]).

Proof. Let $u \in \dot{W}^{1,2}(\mathbb{R}^n_+)$ satisfy the conditions of Theorem 4.1.41 for $f \in C_c^\infty(\mathbb{R}^n_+)$. We know (see, e.g., Chapter 8 in [9]) that (4.1.42) holds when $f \equiv 0$. That means that for

$$v(x) := u(x) - \int_{\mathbb{R}^n_+} f(y)\Gamma(x;y)dy, \quad x \in \mathbb{R}^n_+,$$

we have

$$v(x) = \mathcal{S}_t(\partial_\nu^+ v)(x') - \mathcal{D}_t(\mathrm{Tr}\ v)(x').$$

Note that $v \in \dot{W}^{1,2}(\mathbb{R}^n_+)$ since $u \in \dot{W}^{1,2}(\mathbb{R}^n_+)$ and $f \in C_0^\infty(\mathbb{R}^n_+)$ so that

$$\int_{\mathbb{R}^n_+} f(y)\Gamma(x;y)dy \in Y^{1,2}(\mathbb{R}^n_+) \hookrightarrow \dot{W}^{1,2}(\mathbb{R}^n_+)$$

by Theorem 3.2.6 and Lemma 2.3.7. It remains to prove that for

$$w(x) := \int_{\mathbb{R}^n_+} f(y)\Gamma(x;y)\,dy, \quad x \in \mathbb{R}^n_+,$$

we have

$$\mathcal{S}_t(\partial_\nu^+ w)(x') - \mathcal{D}_t(\mathrm{Tr}\ w)(x') = 0, \quad (x,t) \in \mathbb{R}^n_+.$$

Indeed, for $x = (x', t) \in \mathbb{R}^n_+$,

$$\begin{aligned}
&\mathcal{S}_t(\partial^+_\nu w)(x') - \mathcal{D}_t(\mathrm{Tr}\ w)(x') \\
&\quad = \mathcal{S}_t\left(\int_{\mathbb{R}^n_+} f(y)\partial^+_{\nu_x}\Gamma(\,\cdot\,,0;y)\,dy\right)(x') - \mathcal{D}_t\left(\int_{\mathbb{R}^n_+} f(y)\Gamma(\,\cdot\,,0;y)\,dy\right)(x') \\
&\qquad = \int_{\mathbb{R}^{n-1}} \Gamma(x',t;z',0)\int_{\mathbb{R}^n_+} f(y)\partial^+_{\nu_z}\Gamma(z',0;y)\,dydz' \\
&\qquad - \int_{\mathbb{R}^{n-1}} \overline{\left(\partial^+_{\nu^*_z}\Gamma^*(\,\cdot\,,\,\cdot\,;x',t)\right)}(z',0)\int_{\mathbb{R}^n_+} f(y)\Gamma(z',0;y)\,dydz' \\
&\qquad = -\int_{\mathbb{R}^{n-1}} \Gamma(x',t;z',0)\int_{\mathbb{R}^n_+} f(y)\partial^-_{\nu_z}\Gamma(z',0;y)\,dydz' \\
&\qquad + \int_{\mathbb{R}^{n-1}} \overline{\left(\partial^-_{\nu^*_z}\Gamma^*(\,\cdot\,,\,\cdot\,;x',t)\right)}(z',0)\int_{\mathbb{R}^n_+} f(y)\Gamma(z',0;y)\,dydz'
\end{aligned}$$

Here, all normal derivatives are interpreted in the weak sense and we have used (4.0.37) in the last step. We observe that $\Gamma(\,\cdot\,;y)$ is in $\dot{W}^{1,2}$ away from the pole (it belongs to $Y^{1,2}(\mathbb{R}^n \setminus B_r(y))$ according to the construction of [27]). Thus, we can write the last expression in the display above, by definition of conormal derivative, as

$$\begin{aligned}
&-\int_{\mathbb{R}^n_-}\int_{\mathbb{R}^n_+} \nabla_z\Gamma(x;z)A(z)\nabla_z\Gamma(z;y)f(y)\,dydz \\
&\qquad\qquad + \int_{\mathbb{R}^n_-}\int_{\mathbb{R}^n_+} \overline{A^*(z)}\nabla_z\Gamma(x;z)\nabla_z\Gamma(z;y)f(y)\,dydz = 0,
\end{aligned}$$

as desired, where in the last step we have used that $\overline{\Gamma^*(z;x)} = \Gamma(x;z)$. Similar arguments hold in the lower half-space. $\square$

Theorem 4.1.43. *$\partial^+_\nu\Gamma(\,\cdot\,,0;y)$ and $\nabla_{||}\Gamma(\,\cdot\,,0;y)$ are in $H^{1,\infty}(\mathbb{R}^{n-1})$ uniformly in $y \in \mathbb{R}^n_+$.*

Proof. Define

$$H_\epsilon(x) := (\Phi_\epsilon * \Gamma(x;\,\cdot\,))(y) - (\Phi_\epsilon * \Gamma(x;\,\cdot\,))(y^*),$$

where $\{\Phi_\epsilon\}_{\epsilon>0}$ is a standard set of mollifiers. Fix $x_0 = (x_0', t_0) \in \mathbb{R}^n_+$. For sufficiently small ϵ, we have by Theorem 4.1.41,

$$\begin{aligned} H_\epsilon(x_0) &= -\int_{\mathbb{R}^n_-} \Phi_\epsilon(y^* - z)\Gamma(x_0; z)dz - \mathcal{D}_{t_0}(\mathrm{Tr}\ H_\epsilon)(x_0') + \mathcal{S}_{t_0}(\partial_\nu^- H_\epsilon)(x_0') \\ &= -(\Phi_\epsilon * \Gamma(x_0;\,\cdot\,))(y^*) - \mathcal{D}_{t_0}(\mathrm{Tr}\ H_\epsilon)(x_0') + \mathcal{S}_{t_0}(\partial_\nu^- H_\epsilon)(x_0') \end{aligned}$$

Adding $(\Phi_\epsilon * \Gamma(x_0;\,\cdot\,))(y^*)$ to both sides gives

$$(\Phi_\epsilon * \Gamma(x_0;\,\cdot\,))(y) = -\mathcal{D}_{t_0}(\mathrm{Tr}\ H_\epsilon)(x_0') + \mathcal{S}_{t_0}(\partial_\nu^- H_\epsilon)(x_0').$$

Taking ϵ to zero, we see that for $x = (x', t) \in \mathbb{R}^n_+$,

$$\Gamma(x, y) = -\mathcal{D}_t\left(\mathrm{Tr}\ \widetilde{\Gamma}_y\right)(x') + \mathcal{S}_t\left(\partial_\nu^- \widetilde{\Gamma}_y\right)(x'),$$

thus,

$$\partial_\nu^+ \Gamma(x', 0; y) = -\partial_\nu^+ \mathcal{D}\left(\mathrm{Tr}\ \widetilde{\Gamma}_y\right)(x') + \left(\frac{1}{2}I + \widetilde{K}\right)\left(\partial_\nu^- \widetilde{\Gamma}_y\right)(x'),$$

and

$$\nabla_{x'}\Gamma(x', 0; y) = \nabla_{x'}\left(\frac{1}{2}I - K\right)\left(\mathrm{Tr}\ \widetilde{\Gamma}_y\right)(x') + \nabla_{x'}\mathcal{S}_0\left(\partial_\nu^- \widetilde{\Gamma}_y\right)(x').$$

Now the result follows from Theorems 4.1.35, 4.1.17, and 4.1.13 and the fact that $L^1 \hookrightarrow H^{1,\infty}$. □

4.2 Proofs of main results for homogeneous operators

Proof of Theorem 4.0.22. We aim to employ formulas (4.0.20) and (4.0.21). In what follows we shall discuss how the mapping properties of the layer potentials proved above allow to make sense of these formulas and to establish membership of the gradient of the Green function to the corresponding weak-type space. In the presence of *compatible* invertibility the formulas make sense right away in $\dot{W}^{1,2}(\mathbb{R}^n_+)$ and yield the Green function of Theorem 3.2.10 and Corrolary 3.2.12. Thus, we only need to show that $\nabla_x G_D(\,\cdot\,, y)$, $\nabla_x G_N(\,\cdot\,, y) \in L^{\frac{n+1}{n},\infty}(\mathbb{R}^n_+)$ uniformly in $y \in \mathbb{R}^n_+$.

Applying ∇ to (4.0.20) and (4.0.21) then utilizing Theorem 4.1.43, we see that it suffices to show

1. $\nabla\Gamma(\,\cdot\,; y) \in L^{\frac{n+1}{n},\infty}(\mathbb{R}^n_+)$ uniformly in $y \in \mathbb{R}^n_+$ and

2. $\nabla\mathcal{S}_t : H^{1,\infty}(\mathbb{R}^n) \to L^{\frac{n+1}{n},\infty}(\mathbb{R}^n_+)$ is bounded.

(1) is precisely (3.2.59)

Turning to (2), [28, Lemma 2.2] states that if $w \in L^2_{\text{loc}}(\mathbb{R}^n_+)$ and $\widetilde{\mathcal{N}}(w) \in L^p(\mathbb{R}^n)$ for $0 < p \le \frac{2n}{n+1}$, then

$$w \in L^{\frac{(n+1)p}{n}}(\mathbb{R}^n_+) \quad \text{and} \quad ||w||_{L^{\frac{(n+1)p}{n}}(\mathbb{R}^n_+)} \le C||\widetilde{\mathcal{N}}(w)||_{L^p(\mathbb{R}^n)}, \tag{4.2.1}$$

where C depends only on the domain and on p.

For $f \in H^p(\mathbb{R}^n)$, it follows from (4.1.8) and (4.1.7) that $\nabla\mathcal{S}_t f \in L^2_{\text{loc}}(\mathbb{R}^n_+)$ and $\widetilde{\mathcal{N}}(\nabla\mathcal{S}_t f) \in L^p(\mathbb{R}^n_+)$ so that (4.2.1) applies for $w = \nabla\mathcal{S}_t f$. Thus, (4.1.7) and (4.2.1) give the following bounds:

$$||\nabla\mathcal{S}_t f||_{L^{\frac{(n+1)p}{n}}(\mathbb{R}^n_+)} \le C||\widetilde{\mathcal{N}}(\nabla\mathcal{S}_t f)||_{L^p(\mathbb{R}^n)} \le C||f||_{H^p(\mathbb{R}^n)}$$

for $p \in (p_\alpha, \frac{2n}{n+1})$. Therefore, $\nabla\mathcal{S}_t : H^p(\mathbb{R}^n) \to L^{\frac{(n+1)p}{n}}(\mathbb{R}^n_+)$ is a bounded operator for all such p. Recall that $p_\alpha < 1$, so by interpolation, $\nabla\mathcal{S}_t : H^{1,\infty}(\mathbb{R}^n) \to L^{\frac{n+1}{n},\infty}(\mathbb{R}^n_+)$ is bounded. $\square$

Proof of Corollary 4.0.27. We will prove the result for operator $\mathbb{G}_D$; the proof for $\mathbb{G}_N$ is exactly the same. We have from (2.3.34)

$$\Big(L^{n+1,1}(\mathbb{R}^n_+)\Big)^* = L^{\frac{n+1}{n},\infty}(\mathbb{R}^n_+).$$

Thus, Theorem 4.0.22 gives us $\nabla_x G_D(\,\cdot\,; y) \in \Big(L^{n+1,1}(\mathbb{R}^n_+)\Big)^*$ uniformly in $y \in \mathbb{R}^n_+$. Now,

$$(\nabla\mathbb{G}_D)^* f(y) = \int_{\mathbb{R}^n_+} \overline{\nabla_x G_D(x; y)} f(x)\, dx.$$

Therefore,

$$(\nabla\mathbb{G}_D)^* : L^{n+1,1}(\mathbb{R}^n_+) \to L^\infty(\mathbb{R}^n_+)$$

is linear and bounded.

Thus, the solution operator is bounded as the map

$$\nabla\mathbb{G}_D(\,\cdot\,; y) : \Big(L^\infty(\mathbb{R}^n_+)\Big)^* \to \Big(L^{n+1,1}(\mathbb{R}^n_+)\Big)^* = L^{\frac{n+1}{n},\infty}(\mathbb{R}^n_+).$$

Finally, since $L^1(\mathbb{R}^n_+) \hookrightarrow \left(L^\infty(\mathbb{R}^n_+)\right)^*$,

$$\nabla \mathbb{G}_D : L^1(\mathbb{R}^n_+) \to L^{\frac{n+1}{n},\infty}(\mathbb{R}^n_+)$$

is bounded.

$\square$

Appendix A

Reflection of Solutions

Take $\mathcal{L}$ to be a second order elliptic operator defined on $\mathbb{R}^n_+$, and $\mathcal{B}$ the corresponding bilinear form, as defined in Section 3.1 with $N = 1$. Let $u \in Y^{1,2}(\mathbb{R}^n_+)$ be a solution to $\mathcal{L}u = f$ in $\mathbb{R}^n_+$ with either Dirichlet or Neumann boundary conditions. We seek to extend u to $u^\# \in Y^{1,2}(\mathbb{R}^n)$, and similarly $f^\#$, $\mathcal{L}^\#$, so that $\mathcal{L}^\# u^\# = f^\#$ in $\mathbb{R}^n$.

We extend $\mathcal{L}$ to $\mathcal{L}^\#$, defined on $\mathbb{R}^n$, by odd reflection in all coefficients associated to exactly one x_n-derivative and even reflection in all other coefficients. That is, we write

$$\mathcal{L}^\# u = -D_\alpha \left(A^{\#,\alpha\beta} D_\beta u + b^{\#,\alpha} u \right) + d^{\#,\beta} D_\beta u + V^\# u,$$

where on $\mathbb{R}^n_+$, $A^\# = A$, $b^\# = b$, $d^\# = d$, and $V^\# = V$, and on $\mathbb{R}^n_-$ we set

$$\begin{aligned}
A^{\#,\alpha\beta}(x, x_n) &= \begin{cases} A^{\alpha\beta}(x, -x_n) & \alpha, \beta \neq n \text{ or } \alpha = \beta = n \\ -A^{\alpha\beta}(x, -x_n) & \text{otherwise}, \end{cases} \\
b^{\#,\alpha}(x, x_n) &= \begin{cases} b^{\alpha}(x, -x_n) & \alpha \neq n \\ -b^{n}(x, -x_n) & \alpha = n, \end{cases} \\
d^{\#,\beta}(x, x_n) &= \begin{cases} d^{\beta}(x, -x_n) & \beta \neq n \\ -d^{n}(x, -x_n) & \beta = n, \end{cases} \\
V^\#(x, x_n) &= V(x, -x_n).
\end{aligned} \tag{A.0.1}$$

We define the bilinear form associated to $\mathcal{L}^\#$ by

$$\mathcal{B}^\# \left[u, v \right] = \int_{\mathbb{R}^n} A^{\#,\alpha\beta} D_\beta u \cdot D_\alpha v + b^{\#,\alpha} u \cdot D_\alpha v + d^{\#,\beta} D_\beta u \cdot v + V u \cdot v,$$

and we say that $u^{\#} \in Y^{1,2}(\mathbb{R}^n)$ is a solution to $\mathcal{L}^{\#}u^{\#} = f^{\#} \in L^2(\mathbb{R}^n)$ if

$$\mathcal{B}^{\#}\left[u^{\#}, v\right] = \int f^{\#}v, \quad \forall v \in C_c^{\infty}(\mathbb{R}^n). \tag{A.0.2}$$

First, we prove that the reflected functions exist in the appropriate function spaces.

Lemma A.0.1. *Let $u \in Y_0^{1,2}(\mathbb{R}^n_+)$. Define $u^{\#}$ by odd reflection – i.e.,*

$$u^{\#}(x', x_n) = \begin{cases} u(x', x_n) & \text{if } x_n \geq 0 \\ -u(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Then, $u^{\#} \in Y^{1,2}(\mathbb{R}^n)$.

Proof. Take $\{\psi_k\} \subset C_c^{\infty}(\mathbb{R}^n_+)$ such that $\psi_k \to u$ in $Y^{1,2}(\mathbb{R}^n_+)$. Define $\psi_k^{\#}$ by odd reflection. Then $\psi_k^{\#} \in C_c^{\infty}(\mathbb{R}^n) \subset Y^{1,2}(\mathbb{R}^n)$ and $\{\psi_k^{\#}\}$ is Cauchy in $Y^{1,2}(\mathbb{R}^n_{\pm})$, and therefore in $Y^{1,2}(\mathbb{R}^n)$. Thus, $\{\psi_k^{\#}\}$ has a convergent subsequence in $Y^{1,2}(\mathbb{R}^n)$ and its limit must be $u^{\#}$. Therefore, $u^{\#} \in Y^{1,2}(\mathbb{R}^n)$.

□

Lemma A.0.2. *Let $u \in Y^{1,2}(\mathbb{R}^n_+)$ and define $u^{\#}$ by even reflection – i.e.,*

$$u^{\#}(x', x_n) = \begin{cases} u(x', x_n) & \text{if } x_n \geq 0 \\ u(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Then, $u^{\#} \in Y^{1,2}(\mathbb{R}^n)$.

Proof. Clearly $u^{\#} \in L^{2^*}(\mathbb{R}^n)$ and $D_\alpha u^{\#} \in L^2(\mathbb{R}^n_{\pm})$ for $\alpha = 1, \ldots, n$. It remains to check that the weak derivative is defined at the boundary. Let $\phi \in C_c^{\infty}(\mathbb{R}^n)$. We need to show

$$\int_{\mathbb{R}^n} \left(D_\alpha u^{\#}\right)\phi = -\int_{\mathbb{R}^n} u^{\#}\left(D_\alpha \phi\right), \quad \alpha = 1, \ldots, n. \tag{A.0.3}$$

Assume that ϕ is evenly reflected across $\{x_n = 0\}$. We have $D_\alpha u^{\#} = D_\alpha u$ in $\mathbb{R}^n_+$, $\alpha = 1, \ldots, n$, and for a.e. $x \in \mathbb{R}^n_-$,

$$D_\alpha u^{\#}(x', x_n) = \begin{cases} D_\alpha u(x', -x_n) & \text{if } \alpha \neq n \\ -D_n u(x', -x_n) & \text{if } \alpha = n. \end{cases}$$

For $\alpha \neq n$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \left(D_\alpha u^{\#}\right) \phi &= \int_{\{x_n<0\}} D_\alpha u(x', -x_n)\phi(x', -x_n)\, dx + \int_{\{x_n>0\}} D_\alpha u(x', x_n)\phi(x', x_n)\, dx \\
&= 2\int_{\{x_n>0\}} D_\alpha u(x', x_n)\phi(x', x_n)\, dx \\
&= -2\int_{\{x_n>0\}} u(x', x_n) D_\alpha \phi(x', x_n)\, dx \\
&= -\int_{\{x_n>0\}} u(x', x_n) D_\alpha \phi(x', x_n)\, dx - \int_{\{x_n<0\}} D_\alpha u(x', -x_n)\phi(x', -x_n)\, dx \\
&= -\int_{\mathbb{R}^n} u^{\#} \left(D_\alpha \phi\right).
\end{aligned}
$$

Note that no boundary term is picked up when applying integration by parts because $\nu = -e_n$ and $\alpha \neq n$.

For $\alpha = n$, we have

$$\int_{\{x_n<0\}} D_n u^{\#}(x', x_n)\phi(x', x_n)\, dx = \int_{\{x_n>0\}} -D_n u(x', x_n)\phi(x', x_n)\, dx,$$

and similarly

$$\int_{\{x_n<0\}} u^{\#}(x', x_n) D_n\phi(x', x_n)\, dx = \int_{\{x_n>0\}} u(x', x_n)\left(-D_n\phi\right)(x', x_n)\, dx.$$

Therefore, both sides of (A.0.3) are zero when $\alpha = n$. Thus, $u^{\#}$ is weakly differentiable in $\mathbb{R}^n$.

Now, suppose that ϕ is not evenly reflected across $\{x_n = 0\}$. Define

$$\phi_1\left(x', x_n\right) = \phi(x', 0)\eta(x_n), \tag{A.0.4}$$

where $\eta \in C_c^\infty((-1,1))$ is evenly reflected, $0 \leq \eta \leq 1$, and $\eta(0) = 1$. Then $\phi_1 \in C_c^\infty(\mathbb{R}^n)$, ϕ_1 is evenly reflected across $\{x_n = 0\}$, $\phi_2 := \phi - \phi_1 \in C_c^\infty(\mathbb{R}^n)$, and $\phi_2(x', 0) = 0$. We have

$$|\phi_2\left(x', x_n\right)| = |\phi_2\left(x', x_n\right) - \phi_2\left(x', 0\right)| \leq \left|\left|\frac{\partial \phi_2}{\partial x_n}\right|\right|_{L^\infty(\mathbb{R}^n)} |t|.$$

We claim that $\phi_2 \in Y_0^{1,2}\left(\mathbb{R}^n_\pm\right)$. We prove this for the upper-half space; the lower half-space is analogous. Consider smooth cutoff functions $\{\zeta_\rho\}_{\rho>0} \subset C^\infty\left(\mathbb{R}^n_+\right)$ such that

$$\zeta_\rho\left(x', x_n\right) = 1 \text{ for } |x_n| > \rho, \quad \zeta_\rho\left(x', x_n\right) = 0 \text{ for } |x_n| \leq \tfrac{\rho}{2}, \quad |D\zeta_\rho| \lesssim \tfrac{1}{\rho}.$$

Then $\phi_2\zeta_\rho \in C_c^\infty\left(\mathbb{R}_+^n\right)$ and we have

$$||\phi_2\zeta_\rho - \phi_2||_{Y^{1,2}\left(\mathbb{R}_+^n\right)} \leq 2\int_{\mathbb{R}_+^n} |D\phi_2|^2|\zeta_\rho - 1|^2 + |D\zeta_\rho|^2|\phi_2|^2 \to 0, \tag{A.0.5}$$

where in the last step we have used that $|\phi_2| \leq C\rho$ on $\operatorname{supp}(\zeta_\rho)$, $|D\zeta_\rho| \lesssim \frac{1}{\rho}$, and $|\operatorname{supp}(D\zeta_\rho)| \to 0$.

By the above decomposition, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\left(D_\alpha u^{\#}\right)\phi &= \int_{\mathbb{R}^n}\left(D_\alpha u^{\#}\right)\phi_1 + \int_{\mathbb{R}^n}\left(D_\alpha u^{\#}\right)\phi_2 \\ &= \int_{\mathbb{R}^n}\left(D_\alpha u^{\#}\right)\phi_1 - \int_{\mathbb{R}^n} u^{\#}\left(D_\alpha\phi_2\right),\end{aligned}$$

where we have used that $\psi_2 \in C_c^\infty\left(\mathbb{R}_\pm^n\right)$ and in the upper- and lower-half spaces, the weak derivative of $u^{\#}$ is inherited directly from u. Since ϕ_1 is evenly reflected, we are done.

□

A.0.1 Dirichlet problem

Let $f \in L^{\frac{2n}{n-2}}\left(\mathbb{R}_+^n\right)$. We say that $u \in Y_0^{1,2}(\mathbb{R}_+^n)$ solves the Dirichlet problem

$$\begin{cases}\mathcal{L}u = f \\ u(x',0) = 0\end{cases} \tag{A.0.6}$$

if

$$\mathcal{B}\left[u,v\right] = \int_{\mathbb{R}_+^n} fv, \quad \forall v \in C_c^\infty(\mathbb{R}_+^n). \tag{A.0.7}$$

Lemma A.0.3. *Let $u \in Y_0^{1,2}(\mathbb{R}_+^n)$ be a solution to* (A.0.6) *for $f \in L^{\frac{2n}{n-2}}(\mathbb{R}_+^n)$. Define $u^{\#} \in Y^{1,2}(\mathbb{R}^n)$ and $f^{\#} \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ by odd reflection. Then $\mathcal{L}^{\#}u^{\#} = f^{\#}$ in $\mathbb{R}^n$ in the sense of* (A.0.2).

Proof. First, assume $v \in C_c^\infty(\mathbb{R}^n)$ is reflected evenly across $\{x_n = 0\}$. For such v and $f^{\#}$ and for a.e. $x \in \mathbb{R}^n$, we have

$$f^{\#}\left(x',x_n\right)v\left(x',x_n\right) = -f^{\#}\left(x',-x_n\right)v\left(x',-x_n\right).$$

Therefore,

$$\int_{\mathbb{R}^n} f^{\#}v = 0.$$

So to prove that (A.0.2) holds in this case, we must show

$$\mathcal{B}^{\#}\left[u^{\#}, v\right] = 0. \tag{A.0.8}$$

Since v is evenly reflected, we have for a.e $x \in \mathbb{R}^n_-$,

$$D_\alpha v(x', x_n) = \begin{cases} D_\alpha v(x', -x_n) & \text{if } \alpha \neq n \\ -D_n v(x', -x_n) & \text{if } \alpha = n. \end{cases} \tag{A.0.9}$$

Since $u^{\#}$ is oddly reflected, we have for a.e $x \in \mathbb{R}^n_-$,

$$D_\beta u^{\#}(x', x_n) = \begin{cases} -D_\beta u(x', -x_n) & \text{if } \beta \neq n \\ D_n u(x', -x_n) & \text{if } \beta = n. \end{cases} \tag{A.0.10}$$

By (A.0.9), (A.0.10), and (A.0.1), we have for a.e. $x \in \mathbb{R}^n_-$,

$$\begin{aligned} \left(A^{\#,\alpha\beta} D_\beta u^{\#} D_\alpha v\right)(x', x_n) &= \begin{cases} \left(A^{\alpha\beta}\left(-D_\beta u\right) D_\alpha v\right)(x', -x_n) & \alpha, \beta \neq n \\ \left(\left(-A^{\alpha n}\right) D_n u \, D_\alpha v\right)(x', -x_n) & \alpha \neq n,\ \beta = n \\ \left(\left(-A^{n\beta}\right)\left(-D_\beta u\right)\left(-D_n v\right)\right)(x', -x_n) & \alpha = n,\ \beta \neq n \\ \left(A^{nn} D_\beta u \left(-D_n v\right)\right)(x', -x_n) & \alpha = \beta = n \end{cases} \\ &= \left(-A^{\alpha\beta} D_\beta u \, D_\alpha v\right)(x', -x_n), \\ \left(b^{\#,\alpha} u^{\#} D_\alpha v\right)(x', x_n) &= \begin{cases} \left(b^\alpha \left(-u\right) D_\alpha v\right)(x', -x_n) & \alpha \neq n \\ \left(\left(-b^n\right)\left(-u\right)\left(-D_n v\right)\right)(x', -x_n) & \alpha = n \end{cases} = \left(-b^\alpha u \, D_\alpha v\right)(x', -x_n), \\ \left(d^{\#,\beta} D_\beta u^{\#} v\right)(x', x_n) &= \begin{cases} \left(d^\beta \left(-D_\beta u\right) v\right)(x', -x_n) & \beta \neq n \\ \left(\left(-d^n\right) D_n u \, v\right)(x', -x_n) & \beta = n \end{cases} = \left(-d^\beta D_\beta u\right)(x', -x_n), \end{aligned}$$

and $\left(V^{\#} u^{\#}\right)(x', x_n) = \left(-V u\right)(x', -x_n)$. Thus, we have

$$\mathcal{B}^{\#}\left[u^{\#}, v\right] = \mathcal{B}\left[u, v\right] - \mathcal{B}\left[u, v\right] = 0.$$

Now, suppose $v \in C_c^\infty(\mathbb{R}^n)$ is not evenly reflected. By the same method as in (A.0.4)-(A.0.5), we can write $v = v_1 + v_2 + v_3$, where $v_1 \in C_c^\infty\left(\mathbb{R}^n\right)$ is evenly reflected, $v_2 \in Y_0^{1,2}\left(\mathbb{R}^n_+\right)$, and $v_3 \in Y_0^{1,2}\left(\mathbb{R}^n_-\right)$. By the previous result for evenly reflected test functions, we have

$$\mathcal{B}^{\#}\left[u^{\#}, v_1\right] = \int_{\mathbb{R}^n} f^{\#} v_1 = 0. \tag{A.0.11}$$

By the assumption that u is a solution on $\mathbb{R}^n_+$ – i.e., (A.0.7) – and by a limiting argument,

$$\mathcal{B}^{\#}\left[u^{\#}, v_2\right] = \mathcal{B}\left[u, v_2\right] = \int_{\mathbb{R}^n_+} f v_2 = \int_{\mathbb{R}^n} f^{\#} v_2. \tag{A.0.12}$$

Define $\widetilde{v_3}\left(x', x_n\right) := v_3\left(x', -x_n\right) \in Y_0^{1,2}\left(\mathbb{R}^n_+\right)$. Then, by the previous computations, we have

$$\mathcal{B}^{\#}\left[u^{\#}, v_3\right] = -\mathcal{B}^{\#}\left[u^{\#}, \widetilde{v_3}\right] = -\int_{\mathbb{R}^n_+} f^{\#} \widetilde{v_3} = \int_{\mathbb{R}^n} f^{\#} v_3. \tag{A.0.13}$$

Now (A.0.7) follows from (A.0.11), (A.0.12), and (A.0.13). □

A.0.2 Neumann problem

We say that $u \in Y^{1,2}(\mathbb{R}^n_+)$ solves the Neumann problem

$$\begin{cases} \mathcal{L}u = f \in L^{\frac{2n}{n-2}}(\mathbb{R}^n_+) \\ -\sum_{\alpha=1}^n \left(A^{n\alpha} D_\alpha u + b^n u\right) = 0 \end{cases} \tag{A.0.14}$$

if

$$\mathcal{B}\left[u, v\right] = \int f v, \quad \forall v \in C_c^\infty(\mathbb{R}^n).$$

Lemma A.0.4. *Let $u \in Y^{1,2}(\mathbb{R}^n_+)$ be a solution to* (A.0.14) *for $f \in L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)$. Define $u^{\#} \in Y^{1,2}(\mathbb{R}^n)$ and $f^{\#} \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ by even reflection. Then $\mathcal{L}^{\#} u^{\#} = f^{\#}$ in $\mathbb{R}^n$ in the sense of* (A.0.2).

Proof. First, assume that v is evenly reflected. We must show that

$$\mathcal{B}\left[u^{\#}, v\right] = 2 \int_{\mathbb{R}^n_+} f v. \tag{A.0.15}$$

Since u is evenly reflected, we have for a.e. $x \in \mathbb{R}^n_-$,

$$D_\beta u^{\#}(x', x_n) = \begin{cases} D_\beta u(x', -x_n) & \text{if } \beta \neq n \\ -D_n u(x', -x_n) & \text{if } \beta = n. \end{cases} \tag{A.0.16}$$

Computing as before, we find

$$\begin{aligned} \left(A^{\#,\alpha\beta} D_\beta u^{\#} D_\alpha v\right)\left(x', x_n\right) &= \left(A^{\alpha\beta} D_\beta u\, D_\alpha v\right)\left(x', -x_n\right), \\ \left(b^{\#,\alpha} u^{\#} D_\alpha v\right)\left(x', x_n\right) &= \left(b^\alpha u\, D_\alpha v\right)\left(x', -x_n\right), \\ \left(d^{\#,\beta} D_\beta u^{\#} v\right)\left(x', x_n\right) &= \left(d^\beta D_\beta u\right)\left(x', -x_n\right), \end{aligned}$$

and $\left(V^{\#}u^{\#}\right)(x', x_n) = (Vu)(x', -x_n)$. Therefore,

$$\mathcal{B}^{\#}\left[u^{\#}, v\right] = \mathcal{B}[u, v] + \mathcal{B}[u, v] = 2\int_{\mathbb{R}^n_+} fv.$$

Now, suppose $v \in C_c^{\infty}(\mathbb{R}^n)$ is not evenly reflected. As in the previous lemma, we write $v = v_1 + v_2 + v_3$, where $v_1 \in C_c^{\infty}(\mathbb{R}^n)$ is evenly reflected, $v_2 \in Y_0^{1,2}\left(\mathbb{R}^n_+\right)$, and $v_3 \in Y_0^{1,2}\left(\mathbb{R}^n_-\right)$. By an analogous argument as in that lemma, we have

$$\mathcal{B}^{\#}\left[u^{\#}, v_2 + v_3\right] = \int_{\mathbb{R}^n} f^{\#}(v_2 + v_3).$$

By the previous computations,

$$\mathcal{B}^{\#}\left[u^{\#}, v_1\right] = \int_{\mathbb{R}^n} f^{\#} v_1.$$

Therefore, $\mathcal{L}^{\#}u^{\#} = f^{\#}$ in $\mathbb{R}^n$ as desired. $\square$

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